A nonlinear elliptic PDE with multiple Hardy-Sobolev critical exponents in $\mathbb{R}^N$

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Abstract

In this paper, we will study the following PDE in $\mathbb{R}^N$ involving multiple Hardy-Sobolev critical exponents:

$$\begin{cases}\Delta u + \sum_{i=1}^{l} \lambda_i \frac{u^{2^*(s_i)-1}}{|x|^{s_i}} + u^{2^*-1} = 0 \text{ in } \mathbb{R}^N,\\ u \in D_0^{1,2}(\mathbb{R}^N),\end{cases}$$

where $0 < s_1 < s_2 < \cdots < s_l < 2, 2^* := \frac{2N}{N-2},\ 2^*(s) := \frac{2(N-s)}{N-2}$ and there exists some $k \in [1, l]$ such that $\lambda_i > 0$ for $1 \leq i \leq k$; $\lambda_i < 0$ for $k+1 \leq i \leq l$. We prove the existence and non-existence of the positive ground state solution. The symmetry and regularity of the least-energy solution are also investigated.

Key words: Caffarelli-Kohn-Nirenberg inequality, Hardy-Sobolev exponent, Ground state solution, Moving plane method.

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1 Introduction

Consider the following problem:

$$\begin{cases} \Delta u + \sum_{i=1}^{l} \lambda_i \dfrac{u^{2^*(s_i)-1}}{|x|^{s_i}} + u^{2^*-1} = 0 \text{ in } \mathbb{R}^N, \\ u(x) > 0 \text{ in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $0 < s_1 < s_2 < \cdots < s_l < 2, \lambda_i \in \mathbb{R}, 2^* := \frac{2N}{N-2}, 2^*(s) := \frac{2(N-s)}{N-2}$. We see that the nonlinearities involving multiple Hardy-Sobolev critical exponents and thus are not homogeneous.

Recall that on the half space $\mathbb{R}^N_+$, Li and Lin consider the following problem in [10]:

$$\begin{cases} \Delta u + \lambda \frac{u^{2^*(s)-1}}{|x|^{s_1}} + \frac{u^{2^*(s_2)-1}}{|x|^{s_2}} = 0 \quad \text{in } \mathbb{R}^N_+, \\ u(x) > 0 \quad \text{in } \mathbb{R}^N_+, \quad u(x) = 0 \text{ on } \partial\mathbb{R}^N_+. \end{cases} \tag{1.2}$$

They show that (1.2) has a least-energy solution $u \in H^1_0(\mathbb{R}^N_+)$ provided that $N \geq 3, 0 < s_2 < s_1 < 2, \lambda \in \mathbb{R}$. An earlier result for the special case $s_2 = 0$ in equation (1.2) is obtained by Hsia, Lin and Wadade in [8]. Also they study the existence of the least-energy solution.

In the current paper, we consider the equation defined in the whole space $\mathbb{R}^N$ with multiple Hardy-Sobolev exponents. It seems that the existence of least energy solution to (1.1) is unknown.

Theorem 1.1. *Let $N \geq 3, 0 < s_1 < s_2 < \cdots < s_l < 2$. Suppose that there exists some $k \in [1, l]$ such that $\lambda_i > 0$ for $1 \leq i \leq k$ and $\lambda_i < 0$ for $k+1 \leq i \leq l$. Furthermore, if $N = 3$ and $k \neq l$, we assume that either $s_1 < 1$ or $1 \leq s_1 < 2$ along with $\max\{|\lambda_{k+1}|, \cdots, |\lambda_l|\}$ small enough. Then the following problem*

$$\begin{cases} \Delta u + \sum_{i=1}^{l} \lambda_i \dfrac{u^{2^*(s_i)-1}}{|x|^{s_i}} + u^{2^*-1} = 0 \textit{ in } \mathbb{R}^N, \\ u(x) > 0 \textit{ in } \mathbb{R}^N, \end{cases} \tag{1.3}$$

has a least-energy solution $u(x)$. Moreover, $u(x)$ is radial symmetric decreasing and there exists a constant $C > 0$ such that $u(x) \leq C(1+|x|^{2-N})$ and $|\nabla u(x)| \leq C(1 + |x|^{1-N})$, $0 < \lim_{x\to 0} u(x) = \sup_{x\in\mathbb{R}^N} u(x) < \infty$.

Remark 1.1. *When all λ_is are negative, we claim that there is no least energy solution to equation (1.3).*

Next we prove this claim. The energy functional corresponding to problem (1.3) is defined by

$$\Phi(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \sum_{i=1}^{l} \frac{1}{2^*(s_i)}\lambda_i \int_{\mathbb{R}^N} \frac{|u|^{2^*(s_1)}}{|x|^{s_1}} dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx \tag{1.4}$$

for $u \in D_0^{1,2}(\mathbb{R}^N)$. It is easy to see that $\Phi(u)$ satisfies the mountain pass geometric structure. Define

$$c_* := \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \Phi(\gamma(t)), \tag{1.5}$$

where $\Gamma := \Big\{\gamma(t) \in C\big([0,1], D_0^{1,2}(\mathbb{R}^N)\big) : \gamma(0) = 0, \gamma(1) = u_0\Big\}$ and $\Phi(u_0) < 0$. Then by the mountain pass lemma, the $(PS)_{c_*}$ sequence exists. However, since the embedding $D_0^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*(s)}\left(\mathbb{R}^N, \dfrac{dx}{|x|^s}\right)$ is not compact for $s \in [0,2]$, Φ does not satisfy the Palais-Smale condition. Hence, c_* may not be a critical value of Φ. Indeed, if all λ_is are negative, the constant c_* is always equal to $\frac{1}{N}S^{\frac{N}{2}}$, where S is the best constant of the Sobolev embedding. To see this, we note that the map $t \mapsto \Phi(tu)$ for fixed $u \neq 0$ has a unique maximum for all $t > 0$. It follows that

$$c_* = \inf_{u\in D_0^{1,2}(\mathbb{R}^N)\setminus\{0\}} \max_{t\geq 0} \Phi(tu). \tag{1.6}$$

Denote $\Psi(u) := \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^N} |\nabla u|^2 dx t^2 - \dfrac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx t^{2^*}$ for all $u \in D_0^{1,2}(\mathbb{R}^N)$. Then it is easy to see that

$$\begin{aligned} c_* &= \inf_{u\in D_0^{1,2}(\mathbb{R}^N)\setminus\{0\}} \max_{t>0}\ \Phi(tu) \\ &\geq \inf_{u\in D_0^{1,2}(\mathbb{R}^N)\setminus\{0\}} \max_{t>0}\ \Psi(tu) \\ &= \frac{1}{N}S^{\frac{N}{2}}. \end{aligned} \tag{1.7}$$

On the other hand, recalling that the instanton $U(x) := \dfrac{[N(N-2)]^{\frac{N-2}{4}}}{[1+|x|^2]^{\frac{N-2}{2}}}$ is a minimizer for S. Then we have $\|U\|^2 = S|U|_{2^*}^2 = S^{\frac{N}{2}}$. Noting that for any $x_0 \in \mathbb{R}^N$, $U(x - x_0)$ is also a minimizer for S. Now, let $0 \neq x_0 \in \mathbb{R}^N$, and $\psi \in D_0^{1,2}(\mathbb{R}^N)$ be a nonnegative function such that $\psi \equiv 1$ on $B(0,\rho), 0 < \rho < |x_0|$. For $\varepsilon > 0$, we define

$$U_\varepsilon(x) := \varepsilon^{\frac{2-N}{2}} U(\frac{x - x_0}{\varepsilon}), u_\varepsilon(x) := \psi(x - x_0)U_\varepsilon(x).$$

A direct computation shows that $\lim\limits_{\varepsilon\to 0} \max\limits_{t>0} \Phi(tu_\varepsilon) = \dfrac{1}{N}S^{\frac{N}{2}}$, it follows that

$$c_* \leq \frac{1}{N}S^{\frac{N}{2}}. \tag{1.8}$$

Combine (1.7) and (1.8), we see that $c_* = \frac{1}{N}S^{\frac{N}{2}}$. Hence c_* is never a critical value for Φ when all λ_is are negative, i.e., there are no least energy solutions if all λ_is are negative. To see this, one can assume that $V \in D_0^{1,2}(\mathbb{R}^N)$ such that

$\Phi(V) = \max_{t>0} \Phi(tV) = \frac{1}{N} S^{\frac{N}{2}}$. *Let* $t^* > 0$ *such that* $\Psi(t^*V) = \max_{t>0} \Psi(tV)$. *Then we have*

$$\max_{t>0} \Psi(tV) = \Psi(t^*V) < \Phi(t^*V) \leq \max_{t>0} \Phi(tV) = \Phi(V) = \frac{1}{N} S^{\frac{N}{2}}.$$

Take $W = t^*V$, *then we have*

$$\int_{\mathbb{R}^N} |\nabla W|^2 dx = \int_{\mathbb{R}^N} |W|^{2^*} dx \text{ and } \Psi(W) = \frac{1}{N} \int_{\mathbb{R}^N} |\nabla W|^2 dx < \frac{1}{N} S^{\frac{N}{2}}.$$

It follows that

$$\frac{\int_{\mathbb{R}^N} |\nabla W|^2 dx}{\left(\int_{\mathbb{R}^N} |W|^{2^*} dx\right)^{\frac{2}{2^*}}} = \left(\int_{\mathbb{R}^N} |\nabla W|^2 dx\right)^{1-\frac{2}{2^*}} < \left(S^{\frac{N}{2}}\right)^{\frac{2}{N}} = S,$$

a contradiction to the definition of S.

Thus, in the present paper, we always assume that $k \neq 0$. Besides, we may observe some different behaviors between $k = l$ and $k \neq l$.

Remark 1.2. *Note that when* $s_2 > 0$, *we have* $2^*(s_2) - 1 < 2^* - 1$. *Then the subcritical equation*

$$\Delta \tilde{v} + \tilde{v}^{2^*(s_2)-1} = 0 \text{ in } \Omega \tag{1.9}$$

has no nontrivial solution if $\Omega = \mathbb{R}^N_+$. *This result plays a crucial role in [10]. When we consider the domain* $\Omega = \mathbb{R}^N$, *if* $s_2 > 0$, *we see that* (1.9) *also has no nontrivial solution. For this case, one can modify Li and Lin's arguments in [10] if* $0 < s_2 < s_1 < 2$, *and obtain the existence of ground state solution to problem*

$$\Delta u + \lambda \frac{u^{2^*(s)-1}}{|x|^{s_1}} + \frac{u^{2^*(s_2)-1}}{|x|^{s_2}} = 0, x \in \mathbb{R}^N, 0 < u \in D_0^{1,2}(\mathbb{R}^N).$$

Note that this phenomenon will change essentially when $\Omega = \mathbb{R}^N$ *and* $s_2 = 0$. *Since in this case,* (1.9) *possesses a positive solution. Hence, when applying the blow-up method, ones need a further detailed arguments on the energy to deduce a contradiction. However, if we consider the problem* (1.3) *with* $l > 1$, *i.e., the nonlinearities consist of multiple Hardy-Sobolev critical terms, the arguments of [8] can not be applied directly to study the equation (1.3). Especially, when* $k \neq l$, *their arguments will fail.*

Remark 1.3. *For the case of* $k = l$ *in Theorem 1.1, i.e., all* $\lambda_i s$ *are positive, the ideas of studying the existence of positive ground state solution can be described as following, which is a kind of developing the ideas of Lions [12]. Firstly, we will choose a sequence* $\{u_\varepsilon\}$ *which is a positive ground state solution to a suitable approximating problem (see Theorem 4.1). We shall prove that when* u_ε *possesses a nontrivial weak limit up to a subsequence, then the weak limit is a positive ground state solution of* (1.3) *(see Lemma 6.2). Combining with*

the Pohozaev identity, we can prove that the sequence $\{u_\varepsilon\}$ possesses the good property that "vanishing" can not happen, see Corollaries 5.2 and 6.1. Since the non-homogeneousity, we can not prove the strict subadditivity conditions. However, we will establish a sequence of Lemmas to exclude the possibility of "dichotomy", for the details we refer to subsection 6.2.

Remark 1.4. *The case of $k \neq l$ in Theorem 1.1 is much more complicated. It is not easy to exclude the"vanishing" phenomenon. We will apply a different method to study this case by considering the following variant problem:*

$$\begin{cases} \Delta u + u^{2^*-1} + \sum_{i=1}^{k} \lambda_i \frac{u^{2^*(s_i)-1}}{|x|^{s_i}} + \lambda \left(\sum_{i=k+1}^{l} |\lambda_i| \frac{u^{2^*(s_i)-1}}{|x|^{s_i}} \right) = 0, \ x \in \ \mathbb{R}^N, \\ u \in D_0^{1,2}(\mathbb{R}^N). \end{cases} \tag{1.10}$$

Denote

$$D_k := \{\mu \in \mathbb{R} \ : \ \text{problem (1.10) possesses a least energy solution when } \lambda = \mu\}. \tag{1.11}$$

We shall prove $-1 \in D_k$. Basing on the regularity and symmetry of positive solution established in section 2 and section 3, we will apply the perturbation argument to deduce that $\emptyset \neq D_k$ is a set both open and closed. Thus $D_k = \mathbb{R}$, and it follows that $-1 \in D_k$, which completes the proof.

Remark 1.5. *Unlike the ideas applied in [10] and [8], where the approximation problems are defined in a bounded domain of $\mathbb{R}^N$ hence the approximation sequence has the same bounded support in $\mathbb{R}^N$, our approximation scheme is defined in the whole space $\mathbb{R}^N$ when studying the case of $k = l$. On the other hand, when applying the blow-up method in [10] and [8], they have to prove a new re-scaled sequence $\bar{v}_\varepsilon \to v \neq 0$ in $C^2_{loc}(\mathbb{R}^N)$, as well as the support of $\bar{v}_\varepsilon$ can be expanded to the whole space $\mathbb{R}^N$. However, in the current article, we only need to show that the approximation sequence possesses a nontrivial weak limit.*

Remark 1.6. *We remark that there are some works on the Hardy-Sobolev critical elliptic equations with boundary singularities and on the effect of curvature for the best constant in the Hardy–Sobolev inequalities, see [4, 5, 6, 8, 10, 11] and the references therein. The limiting equations of [4, 5, 6, 8, 10, 11] are actually the form of (1.1) defined in a cone.*

This paper is organized as follows. In section 2, we will study the regularity of the nonnegative solution of (1.3). In section 3, we will study the symmetry of the positive solutions of (1.3). And in section 4 we will firstly study an approximating problem of (1.3). In section 5, we will introduce some interpolation inequalities and the Pohozaev identity for such equation. Finally, in section 6, we will prove the existence of ground state solution and complete the proof of Theorem 1.1.

2 The regularity of the solution to equation (1.1)

Proposition 2.1. *Let $N \geq 3, 0 < s_i < 2, \lambda_i \neq 0, i = 1, 2, \cdots, l$ and set $s_0 = 0, \lambda_0 = 1$. Then any nonnegative $D_0^{1,2}(\mathbb{R}^N)-$solution u of*

$$\int_{\mathbb{R}^N} \nabla u \nabla \varphi dx = \sum_{i=0}^{l} \lambda_i \int_{\mathbb{R}^N} \frac{u^{2^*(s_i)-1}\varphi}{|x|^{s_i}} dx \quad \forall\, \varphi \in D_0^{1,2}(\mathbb{R}^N) \tag{2.1}$$

is of class $L^\infty_{loc}(\mathbb{R}^N)$.

Proof. Let χ be a cut-off function in a ball $\mathbb{B}_R(x_0)$. We take $\varphi = \chi^2 u u_M^{2(t-1)}$, where $t > 1, M > 1$ and $u_M := \min\{u, M\}$. Note that $\nabla u \nabla u_M = |\nabla u_M|^2$ and $\nabla u \nabla u_M u u_M^{2(t-1)-1} = |\nabla u_M|^2 u_M^{2(t-1)}$. Then by (2.1), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \nabla u \nabla \varphi dx \\
=& \int_{\mathbb{R}^N} \nabla u \cdot \left[2\chi \nabla \chi u u_M^{2(t-1)} + \chi^2 u_M^{2(t-1)} \nabla u + 2(t-1)\chi^2 u u_M^{2(t-1)-1} \nabla u_M\right] \\
=& \int_{\mathbb{R}^N} \chi^2 |\nabla u|^2 u_M^{2(t-1)} dx + 2(t-1) \int_{\mathbb{R}^N} \chi^2 |\nabla u_M|^2 u_M^{2(t-1)} dx \\
&+ 2 \int_{\mathbb{R}^N} \chi u u_M^{2(t-1)} \nabla \chi \nabla u dx \\
=& \sum_{i=0}^{l} \lambda_i \int_{\mathbb{R}^N} \frac{u^{2^*(s_i)}}{|x|^{s_i}} \chi^2 u_M^{2(t-1)} dx.
\end{aligned} \tag{2.2}$$

By the Young's inequality, we have

$$|(\chi \nabla u) \cdot (u \nabla \chi)| \leq |\nabla \chi|^2 u^2 + \frac{1}{4} \chi^2 |\nabla u|^2. \tag{2.3}$$

Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^N} \chi^2 |\nabla u|^2 u_M^{2(t-1)} dx + 2(t-1) \int_{\mathbb{R}^N} \chi^2 |\nabla u_M|^2 u_M^{2(t-1)} dx \\
\leq& \sum_{i=0}^{l} \lambda_i \int_{\mathbb{R}^N} \frac{u^{2^*(s_i)}}{|x|^{s_i}} \chi^2 u_M^{2(t-1)} dx + 2 \int_{\mathbb{R}^N} u_M^{2(t-1)} \left[|\nabla \chi|^2 u^2 + \frac{1}{4} \chi^2 |\nabla u|^2\right] dx
\end{aligned} \tag{2.4}$$

and it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \chi^2 |\nabla u|^2 u_M^{2(t-1)} dx + 4(t-1) \int_{\mathbb{R}^N} \chi^2 |\nabla u_M|^2 u_M^{2(t-1)} dx \\
\leq& 2 \sum_{i=0}^{l} \lambda_i \int_{\mathbb{R}^N} \frac{u^{2^*(s_i)}}{|x|^{s_i}} \chi^2 u_M^{2(t-1)} dx + 4 \int_{\mathbb{R}^N} u_M^{2(t-1)} |\nabla \chi|^2 u^2 dx.
\end{aligned} \tag{2.5}$$

Now, we take $t = \frac{2^*(\overline{s})}{2} > 1$ with $\overline{s} := \max\{s_0, s_1, \cdots, s_l\} = s_l < 2$ for simplicity. Consider $w_M := \chi u u_M^{t-1}$, by the Hölder inequality and the Hardy-Sobolev inequality, formula (2.5) yields that

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^N} \frac{1}{|x|^{s_i}} (w_M)^{2^*(\overline{s})} dx\right)^{\frac{2}{2^*(\overline{s})}} \\
&\leq C \left(\int_{\mathbb{R}^N} \frac{1}{|x|^{s_i}} (w_M)^{2^*(s_i)} dx\right)^{\frac{2}{2^*(s_i)}} \\
&\leq C \int_{\mathbb{R}^N} |\nabla w_M|^2 dx \\
&\leq C_1 \Big[\int_{\mathbb{R}^N} |\nabla \chi|^2 u^2 u_M^{2(t-1)} dx + \int_{\mathbb{R}^N} \chi^2 |\nabla u|^2 u_M^{2(t-1)} dx + \\
&\quad (t-1)^2 \int_{\mathbb{R}^N} \chi^2 u_M^{2(t-1)} |\nabla u_M|^2 dx \Big] \\
&\leq C_2 t \Big[\sum_{i=0}^{l} |\lambda_i| \int_{\mathbb{R}^N} \frac{u^{2^*(s_i)}}{|x|^{s_i}} \chi^2 u_M^{2(t-1)} dx + \int_{\mathbb{R}^N} u_M^{2(t-1)} |\nabla \chi|^2 u^2 dx \Big].
\end{aligned}
\tag{2.6}
$$

By the Hölder inequality,

$$
\begin{aligned}
&\left| \lambda_i \int_{\mathbb{R}^N} \frac{u^{2^*(s_i)}}{|x|^{s_i}} \chi^2 u_M^{2(t-1)} dx \right| \\
\leq& |\lambda_i| \left[\int_{\mathbb{B}_{R_0}(x_0)} \frac{u^{2^*(s_i)}}{|x|^{s_i}} dx \right]^{\frac{2^*(s_i)-2}{2^*(s_i)}} \cdot \left[\int_{\mathbb{R}^N} \frac{1}{|x|^{s_i}} \left(\chi u u_M^{t-1}\right)^{2^*(s_i)} dx \right]^{\frac{2}{2^*(s_i)}},
\end{aligned}
\tag{2.7}
$$

then by the absolute continuity of the integral, we see that there exists some $R_0 > 0$ small enough such that

$$
\frac{2^*(\overline{s})}{2} C_2 |\lambda_i| \left[\int_{\mathbb{B}_{R_0}(x_0)} \frac{u^{2^*(s_i)}}{|x|^{s_i}} dx \right]^{\frac{2^*(s_i)-2}{2^*(s_i)}} < \frac{1}{l+2} \text{ for all } i = 0, 1, 2, \cdots, l. \tag{2.8}
$$

Hence, for such R_0, we have

$$
\begin{aligned}
&\left(\int_{\mathbb{R}^N} \frac{1}{|x|^{s_i}} (w_M)^{2^*(s_i)} dx\right)^{\frac{2}{2^*(s_i)}} \\
\leq& \frac{1}{l+2} \sum_{j=0}^{l} \left(\int_{\mathbb{R}^N} \frac{1}{|x|^{s_j}} (w_M)^{2^*(s_j)} dx\right)^{\frac{2}{2^*(s_j)}} + \tilde{C}_2 \int_{\mathbb{R}^N} |\nabla \chi|^2 u^{2^*(\overline{s})} dx
\end{aligned}
\tag{2.9}
$$

for all $i = 0, 1, \cdots, l$. It follows that

$$
\sum_{i=0}^{l} \left(\int_{\mathbb{R}^N} \frac{1}{|x|^{s_i}} (w_M)^{2^*(s_i)} dx\right)^{\frac{2}{2^*(s_i)}} \leq C_3 \int_{\mathbb{R}^N} |\nabla \chi|^2 u^{2^*(\overline{s})} dx. \tag{2.10}
$$

Let M go to infinity, we obtain that

$$u \in L^{\frac{2^*(\overline{s})}{2}2^*(s_i)}\left(\mathbb{B}_{\frac{R}{2}}(x_0), \frac{dx}{|x|^{s_i}}\right), \quad i = 0, 1, \cdots, l. \tag{2.11}$$

By the arbitrariness of x_0, we obtain that

$$u \in L_{loc}^{\frac{2^*(\overline{s})}{2}2^*(s_i)}\left(\mathbb{R}^N, \frac{dx}{|x|^{s_i}}\right), \quad i = 0, 1, \cdots, l. \tag{2.12}$$

Now, for any $R > 0, 0 < r < 1$, we take a cut-off function $0 < \chi \leq 1$ in $\mathbb{B}_{R+r}$ such that $\chi \equiv 1$ in $\mathbb{B}_R$ and $|\nabla \chi| \leq \frac{2}{r}$ in $\mathbb{B}_{R+r}$. Set

$$\sigma_i := \frac{2^*(\overline{s})2^*(s_i)}{2[2^*(s_i) - 2]}, \quad i = 0, 1, \cdots, l. \tag{2.13}$$

We note that

$$\frac{2^*(s_i)(\sigma_i - 1)}{2\sigma_i} > 1 \text{ for all } i = 0, 1, \cdots, l, \tag{2.14}$$

we can take proper constants $q_i \leq 2^*(s_i)$ such that

$$\frac{q_i(\sigma_i - 1)}{2\sigma_i} \equiv const > 1. \tag{2.15}$$

By the Hölder inequality, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|x|^{s_i}} u^{2^*(s_i)-2}\chi^2 u_M^{2t} dx \\
\leq &\left[\int_{\mathbb{B}_{R+r}} \frac{1}{|x|^{s_i}} u^{[2^*(s_i)-2]\sigma_i}\chi^2 dx\right]^{\frac{1}{\sigma_i}} \left[\int_{\mathbb{B}_{R+r}} \frac{1}{|x|^{s_i}} \chi^2 u_M^{\frac{2t\sigma_i}{\sigma_i-1}} dx\right]^{\frac{\sigma_i-1}{\sigma_i}} \\
\leq &C_4 \left[\int_{\mathbb{B}_{R+r}} \frac{1}{|x|^{s_i}} \chi^2 u^{\frac{2t\sigma_i}{\sigma_i-1}} dx\right]^{\frac{\sigma_i-1}{\sigma_i}},
\end{aligned} \tag{2.16}$$

provided that $u \in L^{\frac{2t\sigma_i}{\sigma_i-1}}\left(\mathbb{B}_{R+r}, \frac{dx}{|x|^{s_i}}\right)$. Here we remark that by the Höler inequality, C_4 should depend on the volume of the ball $\mathbb{B}_{R+r}$. However, since $r < 1$, we can choose some suitable C_4 that independent of r. Noting that the right hand side of (2.16) is independent of M, by letting M go to infinity, we indeed obtain that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \frac{1}{|x|^{s_i}} u^{2^*(s_i)}\chi^2 u_M^{2(t-1)} dx \leq \int_{\mathbb{R}^N} \frac{1}{|x|^{s_i}} u^{2^*(s_i)}\chi^2 u^{2(t-1)} dx \\
\leq &C_4 \left[\int_{\mathbb{B}_{R+r}} \frac{1}{|x|^{s_i}} u^{\frac{2t\sigma_i}{\sigma_i-1}} dx\right]^{\frac{\sigma_i-1}{\sigma_i}}.
\end{aligned} \tag{2.17}$$

On the other hand,

$$
\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla \chi|^2 u^2 u_M^{2(t-1)} dx \\
&\le \left(\frac{2}{r}\right)^2 \left(\int_{\mathbb{B}_{R+r}} \frac{1}{|x|^{s_i}} u^{\frac{2t\sigma_i}{\sigma_i-1}} dx\right)^{\frac{\sigma-1}{\sigma}} \left(\int_{\mathbb{B}_{R+r}} |x|^{\eta_i} dx\right)^{\frac{1}{\sigma_i}}, \qquad (2.18)
\end{aligned}
$$

where

$$\eta_i = s_i(\sigma_i - 1) \ge 0.$$

Hence,

$$\int_{\mathbb{R}^N} |\nabla \chi|^2 u^2 u_M^{2(t-1)} dx \le C_5 r^{-2} \left(\int_{\mathbb{B}_{R+r}} \frac{1}{|x|^{s_i}} u^{\frac{2t\sigma_i}{\sigma_i-1}} dx\right)^{\frac{\sigma_i-1}{\sigma_i}}. \qquad (2.19)$$

Recalling that $q_i \le 2^*(s_i)$ and $\chi \equiv 1$ in $\mathbb{B}_R$, by the Hölder inequality, inserting (2.17),(2.19) into (2.6) and then letting M go to infinity, we obtain that

$$\sum_{i=0}^{l} \left(\int_{\mathbb{B}_R} \frac{1}{|x|^{s_i}} u^{q_i t} dx\right)^{\frac{1}{q_i t}} \le C_6^{\frac{1}{t}} t^{\frac{1}{2t}} r^{-\frac{1}{t}} \sum_{i=0}^{l} \left(\int_{\mathbb{B}_{R+r}} \frac{1}{|x|^{s_i}} u^{t q_{i,0}} dx\right)^{\frac{1}{t q_{i,0}}}, \quad (2.20)$$

where by (2.15),

$$q_{i,0} := \frac{2\sigma_i}{\sigma_i - 1} < q_i.$$

Recalling (2.15) again, we have that $\tau := \frac{q_i}{q_{i,0}} > 1$ is independent of i. Define $t = \tau^j, R = 1$ and $r_j = 2^{-j}, j \ge 1$, applying iteration, (2.20) yields

$$\sum_{i=0}^{l} \left[\int_{\mathbb{B}_{1+2^{-j+1}}} \frac{1}{|x|^{s_i}} u^{q_i \tau^{j+1}} dx\right]^{\frac{1}{q_i \tau^{j+1}}} \le \prod_{k=0}^{j+1} (C_6 \tau^{\frac{k}{2}} 2^k)^{\tau^{-k}} \sum_{i=0}^{l} \left[\int_{\mathbb{B}_{\frac{3}{2}}} \frac{1}{|x|^{s_i}} u^{q_i} dx\right]^{\frac{1}{q_i}}. \qquad (2.21)$$

Denote

$$\Theta := \prod_{k=0}^{\infty} (C_6 \tau^{\frac{k}{2}} 2^k)^{\tau^{-k}},$$

we have

$$\ln \Theta = \ln C_6 \sum_{k=0}^{\infty} \frac{1}{\tau^k} + \big(\ln 2 + \frac{1}{2} \ln \tau\big) \sum_{k=0}^{\infty} \frac{k}{\tau^k}. \qquad (2.22)$$

It is easy to see $\Theta < \infty$ due to the fact of $\tau > 1$. Hence, letting j go to infinity in (2.21), noting that $s_i \ge 0$, we obtain that

$$\sup_{\mathbb{B}_1} u \le \frac{1}{l+1} \Theta \sum_{i=0}^{l} \left[\int_{\mathbb{B}_{\frac{3}{2}}} \frac{1}{|x|^{s_i}} u^{q_i} dx\right]^{\frac{1}{q_i}}. \qquad (2.23)$$

□

Then we have the following result.

Lemma 2.1. *Let $N \geq 3, 0 < s_i < 2, \lambda_i > 0, i = 1, 2, \cdots, l$. Then any nonnegative solution of*

$$\Delta u + \sum_{i=1}^{l} \lambda_i \frac{u^{2^*(s_i)-1}}{|x|^{s_i}} + u^{2^*-1} = 0 \text{ in } \mathbb{R}^N \backslash \{0\} \tag{2.24}$$

satisfying

$$0 < \liminf_{x \to 0} u(x) \leq \limsup_{x \to 0} u(x) < +\infty. \tag{2.25}$$

Proof. By the standard elliptic estimation, we have that $u \in C^\infty(\mathbb{R}^N) \backslash \{0\}$. Then by [3, Lemma 4.2], take some $r > 0$, we see that $t \mapsto \min_{|x|=t} u(x)$ is concave in t^{2-N} for $s \in (0, r)$. Hence,

$$u(x) \geq \min_{|x|=r} u(x) > 0 \text{ for all } x \in \overline{B_r} \backslash \{0\}, \tag{2.26}$$

and thus

$$\liminf_{x \to 0} u(x) \geq \min_{|x|=r} u(x) > 0. \tag{2.27}$$

On the other hand, by Proposition 2.1, $u(x)$ is of class $L^\infty_{loc}(\mathbb{R}^N)$. Hence, the proof of this lemma is completed. □

Lemma 2.2. *Let $N \geq 3, 0 < s_i < 2, \lambda_i > 0, i = 1, 2, \cdots, l$ and set $s_0 = 0, \lambda_0 = 1$. Then any nonnegative $D_0^{1,2}(\mathbb{R}^N)-$solution of* (2.1) *satisfying*

$$0 < \liminf_{|x| \to \infty} |x|^{N-2} u(x) \leq \limsup_{|x| \to \infty} |x|^{N-2} u(x) < \infty,$$

i.e., $u = O(\frac{1}{|x|^{N-2}})$ when $|x| \to +\infty$.

Proof. When u is a nonnegative solution of (2.1), a direct computation shows that its Kelvin Transform $v(x) := |x|^{-(N-2)} u \left(\frac{x}{|x|^2} \right)$ is also a nonnegative solution of (2.1). Then by Lemma 2.1, we have

$$0 < \liminf_{x \to 0} v(x) \leq \limsup_{x \to 0} v(x) < +\infty,$$

which implies the results of this Lemma. □

Remark 2.1. (*i*) *Indeed, even for the case of $\lambda_i < 0$, if $u(x)$ is a nonnegative solution of* (2.1)*, the corresponding Kelvin Transform $v(x) := |x|^{-(N-2)} u \left(\frac{x}{|x|^2} \right)$ is also a nonnegative solution. Then by the regularity of Proposition 2.1, $|v(x)| \leq C|x|$ for $|x| < 1$. Thus,*

$$|u(x)| \leq C|x|^{2-N} \quad \text{for } |x| \geq 1. \tag{2.28}$$

By Proposition 2.1 again, we can obtain that

$$|u(x)| \leq C\left(1+|x|^{2-N}\right) \text{ for } x \in \mathbb{R}^N. \tag{2.29}$$

And the standard gradient estimate, we obtain

$$|\nabla u(x)| \leq C|x|^{1-N} \text{ for } |x| \geq 1. \tag{2.30}$$

(ii) *Based on the results of section 3 below, we will see that any positive solution is radial symmetric and decreasing by $r = |x|$. Hence, by the monotonicity, $0 < \lim\limits_{x\to 0} u(x) = \sup\limits_{x\in\mathbb{R}^N} u(x) < \infty$ exists and thus, $x = 0$ is a movable singularity (see also Remark 3.2). We also see that the constraint on the sign of λ_is in Lemma 2.1 and Lemma 2.2 can be removed.*

3 The symmetry of the positive solution of equation (1.1)

Remark 3.1. *Without loss of generality, we may assume that $k \neq l$. Indeed, it is easy to see that the arguments in this section are valid for $\lambda_{k+1} = \cdots = \lambda_l = 0$. Which means that our methods are valid for the case of all λ_is being positive.*

In the following, we will apply the well known moving plane method to prove the symmetric property. We start with the Kelvin transform $v(x) = \dfrac{1}{|x|^{N-2}} u\left(\dfrac{x}{|x|^2}\right)$ which has the desired decay rate at infinity as we need. But it has a possible singularity at the origin. We see that $v(x)$ satisfies the following equation:

$$-\Delta v = \sum_{\lambda_i>0} \lambda_i \frac{v^{2^*(s_i)-1}}{|x|^{s_i}} + \sum_{\lambda_i<0} \lambda_i \frac{v^{2^*(s_i)-1}}{|x|^{s_i}}, \quad x \in \mathbb{R}^N \backslash \{0\},\ N \geq 3. \tag{3.1}$$

Here we always assume that $0 \leq s_i < 2$ and strictly increasing by the index i. For any $\lambda \leq 0$, we define

$$\Sigma_\lambda = \left\{x = (x_1, x') \in \mathbb{R}^N \ :\ x_1 < \lambda, x' \in \mathbb{R}^{N-1}\right\},\ T_\lambda = \partial\Sigma_\lambda$$

and let $x_\lambda := (2\lambda - x_1, x')$ be the reflection point of x about the plane T_λ. We also set

$$v_\lambda(x) = v(x^\lambda) \text{ and } w_\lambda(x) := v_\lambda(x) - v(x).$$

We note that for $\lambda < 0$, it always satisfies that

$$|x^\lambda| < |x| \text{ for all } x \in \Sigma_\lambda. \tag{3.2}$$

We are aim to prove that

$$w_\lambda(x) \geq 0 \text{ for all } x \in \Sigma_\lambda. \tag{3.3}$$

Step 1: Move the plane from $-\infty$. Noting that

$$-\Delta v_\lambda(x) = (-\Delta v)\left(x^\lambda\right) = \sum_{\lambda_i>0} \lambda_i \frac{v(x^\lambda)^{2^*(s_i)-1}}{|x^\lambda|^{s_i}} + \sum_{\lambda_i<0} \lambda_i \frac{v(x^\lambda)^{2^*(s_i)-1}}{|x^\lambda|^{s_i}}, \quad (3.4)$$

we obtain that

$$\begin{aligned}
&-\Delta w_\lambda(x) = [-\Delta v_\lambda(x)] - [-\Delta v(x)] \\
&= \sum_{\lambda_i>0} \lambda_i \left[\frac{v\left(x^\lambda\right)^{2^*(s_i)-1}}{|x^\lambda|^{s_i}} - \frac{v(x)^{2^*(s_i)-1}}{|x|^{s_i}} \right] + \sum_{\lambda_i<0} \lambda_i \left[\frac{v\left(x^\lambda\right)^{2^*(s_i)-1}}{|x^\lambda|^{s_i}} - \frac{v(x)^{2^*(s_i)-1}}{|x|^{s_i}} \right] \\
&\geq \sum_{\lambda_i>0} \lambda_i \left[\frac{v\left(x^\lambda\right)^{2^*(s_i)-1}}{|x|^{s_i}} - \frac{v(x)^{2^*(s_i)-1}}{|x|^{s_i}} \right] + \sum_{\lambda_i<0} \lambda_i \left[\frac{v\left(x^\lambda\right)^{2^*(s_i)-1}}{|x^\lambda|^{s_i}} - \frac{v(x)^{2^*(s_i)-1}}{|x^\lambda|^{s_i}} \right].
\end{aligned} \quad (3.5)$$

Here we using the fact of (3.2). By the Mean Value Theorem, it is easy to verify that

$$\begin{aligned}
-\Delta w_\lambda(x) \geq & \left[\sum_{\lambda_i>0} \lambda_i \left[2^*(s_i) - 1\right] \frac{1}{|x|^{s_i}} \left[\psi_\lambda(x)\right]^{2^*(s_i)-2} \right] w_\lambda(x) \\
& + \sum_{\lambda_i<0} \left[\lambda_i \left[2^*(s_i) - 1\right] \frac{1}{|x^\lambda|^{s_i}} \left[\psi_\lambda(x)\right]^{2^*(s_i)-2} \right] w_\lambda(x),
\end{aligned} \quad (3.6)$$

where $\psi_\lambda(x)$ are some number between $v_\lambda(x)$ and $v(x)$. Let

$$\begin{aligned}
c(x) = & -\left[\sum_{\lambda_i>0} \lambda_i \left[2^*(s_i) - 1\right] \frac{1}{|x|^{s_i}} \left[\psi_\lambda(x)\right]^{2^*(s_i)-2} \right] \\
& - \left[\sum_{\lambda_i<0} \lambda_i \left[2^*(s_i) - 1\right] \frac{1}{|x^\lambda|^{s_i}} \left[\psi_\lambda(x)\right]^{2^*(s_i)-2} \right].
\end{aligned} \quad (3.7)$$

By [2, Corollary 7.4.2], we only need to check the decay rate of $c(x)$, and more precisely, only at the points $\tilde{x}$ where w_λ is negative (see [2, Remark 7.4.2]). Apparently at these points

$$v_\lambda(\tilde{x}) < v(\tilde{x})$$

and hence

$$0 \leq v_\lambda(\tilde{x}) \leq \psi_\lambda(x) \leq v(\tilde{x}).$$

Denote

$$d(x) := -\left[\sum_{\lambda_i>0} \lambda_i \left[2^*(s_i) - 1\right] \frac{1}{|x|^{s_i}} \left[\psi_\lambda(x)\right]^{2^*(s_i)-2} \right], \quad (3.8)$$

then at these points, we have

$$-\Delta w_\lambda(x) + d(x) w_\lambda(x) \geq 0. \quad (3.9)$$

A direct computation shows that

$$\psi_\lambda(\tilde{x}) = O\left(\left|\frac{1}{|\tilde{x}|}\right|^{N-2}\right), \tag{3.10}$$

thus we have

$$\frac{[\psi_\lambda(\tilde{x})]^{2^*(s_i)-2}}{|x|^{s_i}} = O\left(\frac{1}{|\tilde{x}|^{4-s_i}}\right). \tag{3.11}$$

Set

$$j = \max\{j \in \{0, 1, \cdots, l\} \ : \ \lambda_j > 0\}, \tag{3.12}$$

then we have

$$d(\tilde{x}) = O\left(\frac{1}{|\tilde{x}|^{4-s_j}}\right). \tag{3.13}$$

Then by $s_j < 2$, we see that the power of $\frac{1}{|\tilde{x}|}$ is greater than two. Therefore, by [2, Theorem 7.4.2], we conclude that for λ sufficiently negative, i.e., $|\tilde{x}|$ sufficiently large, we prove that (3.3) holds and this completes the preparation for the moving of planes.

To continue our argument, we need the following Lemma 3.1 due to [2, Lemma 8.2.1]. Define

$$\bar{w}_\lambda(x) = \frac{w_\lambda(x)}{\phi(x)},$$

where

$$\phi(x) = \frac{1}{|x|^q} \text{ with } 0 < q < N-2.$$

Then we have

$$-\Delta \bar{w}_\lambda = 2\nabla \bar{w}_\lambda \cdot \frac{\nabla \phi}{\phi} + \left(-\Delta w_\lambda + \frac{\Delta \phi}{\phi} w_\lambda\right)\frac{1}{\phi}. \tag{3.14}$$

Then the following result holds.

Lemma 3.1. *(cf.[2, Lemma 8.2.1]) There exists a $R_0 > 0$ (independent of λ), such that if x^o is a minimum point of $\bar{w}_\lambda$ and $\bar{w}_\lambda(x^o) < 0$, then $|x^o| < R_0$.*

Step 2: Move the Plane to the Limiting Position to Derive Symmetry. Now, we let

$$\sigma := \sup\{\lambda \ : \ \lambda \leq 0 \text{ and } w_\lambda \geq 0, \ \forall\, x \in \Sigma_\lambda\}. \tag{3.15}$$

Step 1 yields that $\{\lambda \ : \ \lambda \leq 0 \text{ and } w_\lambda \geq 0, \ \forall\, x \in \Sigma_\lambda\} \neq \emptyset$ and $\sigma > -\infty$. Then $\sigma \leq 0$ and $w_\sigma \geq 0$ for all $x \in \Sigma_\sigma$ and $w_\sigma \equiv 0$ for $x \in \partial\Sigma_\sigma$. We shall prove $\sigma = 0$. By negation, we assume $\sigma < 0$. Then we claim that

$$w_\sigma(x) \equiv 0, \ \forall\, x \in \Sigma_\sigma. \tag{3.16}$$

If not, by the strong maximum principle on unbounded domains with not necessarily non-negative coefficient function (see [2, Theorem 7.3.3]) and the Hopf's Lemma, we obtain that

$$w_\sigma(x) > 0 \text{ in } \Sigma_\sigma \text{ and } \frac{\partial w_\sigma}{\partial \nu} < 0 \text{ on } \partial\Sigma_\sigma = T_\sigma. \tag{3.17}$$

Then by the definition of σ, there exists a decreasing sequence σ_i and a corresponding sequence $\{x^i\}_{i\in\mathbb{N}}$ such that $x^i \in \Sigma_{\sigma_i}, (x^i)_1 < \sigma_i, \lim\limits_{i\to+\infty} \sigma_i = \sigma$ and

$$v_{\sigma_i}(x^i) < v(x^i) \ \text{ i.e., } v\left((x^i)_{\sigma_i}\right) < v(x^i). \tag{3.18}$$

Furthermore, x_i can be chosen as the minimum point of $\bar{w}_{\sigma_i}(x)$. Then $\nabla \bar{w}_{\sigma_i}(x_i) = 0$. By Lemma 3.1, we see that

$$|x^i| \le R_0 \ \forall\ i = 1, 2, \cdots . \tag{3.19}$$

Hence, $\{x^i\}$ is bounded. Then up to a subsequence (still denoted by $\{x^i\}$), we may assume that $\lim\limits_{i\to+\infty} x_i = x \in \overline{\Sigma_\sigma} \cap \{x_1 \le \sigma\}$. Then by (3.18) and $\nabla \bar{w}_{\sigma_i}(x_i) = 0$, we have

$$v_\sigma(x) = v(x_\sigma) \le v(x), \text{ i.e., } w_\sigma(x) \le 0 \text{ and } \nabla \bar{w}_\sigma(x) = 0. \tag{3.20}$$

Then

$$\bar{w}_\sigma(x) \le 0 \text{ by the definition of } \bar{w}_\sigma(x) \text{ and } \phi(x) > 0. \tag{3.21}$$

On the other hand, by (3.17), we obtain the reverse inequality

$$\bar{w}_\sigma(x) \ge 0 \text{ since } w_\sigma(x) \ge 0 \text{ and } \phi(x) > 0. \tag{3.22}$$

Hence, $\bar{w}_\sigma(x) = 0$. Then a direct computation shows that

$$\nabla w_\sigma(x) = \nabla \bar{w}_\sigma(x)\phi(x) + \bar{w}_\sigma(x)\nabla\phi(x) = 0. \tag{3.23}$$

At the same time, we also obtain that $w_\sigma(x) = 0$ and thus $x \in T_\sigma$, i.e., $x_1 = \sigma$. Then by (3.17), we have the outward normal derivative

$$\frac{\partial w_\sigma}{\partial \nu} < 0,$$

a contradiction to (3.23). Thereby the claim (3.16) is proved and then it follows that $v(x)$ is symmetry respect the plain $T_\sigma = \{x_1 = \sigma\}$. Substitute into the equation (3.1), by the fact of $v_\sigma(x) = v(x^\sigma) \equiv v(x)$, we have

$$\sum_{\lambda_i>0} \lambda_i \left[\frac{1}{|x^\sigma|^{s_i}} - \frac{1}{|x|^{s_i}}\right] v(x)^{2^*(s_i)} \equiv -\sum_{\lambda_i<0} \lambda_i \left[\frac{1}{|x^\sigma|^{s_i}} - \frac{1}{|x|^{s_i}}\right] v(x)^{2^*(s_i)}. \tag{3.24}$$

Taking $|x|$ sufficiently large, noting the fact of $|x^\sigma| = |(2\sigma - x_1, x')| = |(x_1 - 2\sigma, x')|$, if $\sigma < 0$, then by the mean value theorem and the decay order of $v(x)$, we obtain that

$$\left[\frac{1}{|x^\sigma|^{s_i}} - \frac{1}{|x|^{s_i}}\right] v(x)^{2^*(s_i)} = O\left(\frac{1}{|x|^{2N-s_i+1}}\right). \tag{3.25}$$

It follows that the order of left hand side is $O\left(\frac{1}{|x|^{2N-s_j+1}}\right)$, where j is given by (3.12). Similarly, we set

$$\tilde{j} = \{i \in \{2, 3, \cdots, l\} \ : \ \lambda_i < 0\} > 1, \tag{3.26}$$

then the right hand side equals $O\left(\frac{1}{|x|^{2N-s_{\tilde{j}}+1}}\right)$. Obviously, $j \neq \tilde{j}$ and it follows that $2N - s_j + 1 \neq 2N - s_{\tilde{j}} + 1$. Hence, if $\sigma < 0$, (3.24) will fail. Thereby, $\sigma = 0$ is proved and it follows that $w_0(x) \geq 0$, i.e.,

$$v(x_1, x') \geq v(-x_1, x') \text{ for all } (x_1, x') \in \mathbb{R}^N \text{ with } x_1 \geq 0. \tag{3.27}$$

We define

$$\tilde{\Sigma}_\lambda = \{x = (x_1, x') \in \mathbb{R}^N \ : \ x_1 > \lambda, x' \in \mathbb{R}^{N-1}\}, \ T_\lambda = \partial\tilde{\Sigma}_\lambda$$

and let

$$\tilde{\sigma} := \inf\{\lambda \ : \ \lambda \geq 0 \text{ and } w_\lambda \geq 0, \ \forall \ x \in \tilde{\Sigma}_\lambda\}. \tag{3.28}$$

Then the similar arguments above can obtain that $\tilde{\sigma} = 0$ and $w_0(x) \geq 0$, i.e.,

$$v(-x_1, x') \geq v(x_1, x') \text{ for all } (x_1, x') \in \mathbb{R}^N \text{ with } x_1 \geq 0. \tag{3.29}$$

Then by (3.27) and (3.29), we obtain that $v(-x_1, x') = v(x_1, x')$ for all $(x_1, x') \in \mathbb{R}^N$. Furthermore, it is easy to see that $v(x_1, x') = v(|x_1|, x')$ is decreasing by $|x_1|$ in $[0, +\infty)$.

Step 3: Prove that $v(x)$ is radially symmetric and monotone decreasing about $x = 0$.
By the results above, we see that $v(x)$ is symmetric with respect to the plane T_0 and decreasing by the distance from T_0. We note that the arguments above are also valid for the any other hyper plane perpendicular to a given versor $\vartheta \neq \overrightarrow{e}_1$. Finally we obtain that $v(x)$ is radially symmetric and monotone decreasing about $x = 0$.

Remark 3.2. *Let $v(x) = v(r)$ be a positive ground state solution to* (1.1), *where $r = |x|$. Firstly, recalling Proposition 2.1, we have that $\limsup_{r\to 0^+} v(r) < \infty$. Secondly, by the radial symmetry and monotone decreasing property, we see that $\lim_{r\to 0^+} v(r)$ exists. Hence, $x = 0$ is a moveable singularity point, i.e., we can define $v(0) = \lim_{r\to 0^+} v(r)$. Moreover, it is easy to see that $v(0) = \sup_{x\in\mathbb{R}^N\setminus\{0\}} v(x)$.*

Now, we can see that the results of Lemmas 2.1 and 2.2 are independent of the sign of λ_is. Up to now, all the results in section 2 and section 3 are independent of the sign of λ_is. Certainly, we do these jobs under the premise of the existence results. In the following sections, we will focus on the existence of least energy solution.

4 Approximating problems

Assume that $0 = s_0 < s_1 < s_2 < \cdots < s_l < 2$. Let $0 < \varepsilon < s_1$ and define

$$a_{i,\varepsilon}(x) := \begin{cases} \frac{1}{|x|^{s_i-\varepsilon}} & \text{for } |x| < 1, \\ \frac{1}{|x|^{s_i+\varepsilon}} & \text{for } |x| \geq 1, \end{cases} \quad i = 1, 2, \cdots, l. \tag{4.1}$$

We also denote $a_{i,0}(x) = \frac{1}{|x|^{s_i}}$. Then it is easy to see that $a_{i,\varepsilon}(x)$ is decreasing with respect to $\varepsilon \in [0, s_1)$.

Lemma 4.1. *Let $0 \leq \varepsilon < s_1$, then for any $u \in D_0^{1,2}(\mathbb{R}^N)$ and $i \in \{1, 2, \cdots, l\}$, $\displaystyle\int_{\mathbb{R}^N} a_{i,\varepsilon}(x)|u|^{2^*(s_i)}dx$ is well defined and decreasing by ε.*

Proof. See [13, Lemma 7.4]. □

Denote by $L^p\left(\mathbb{R}^N, a_{i,\varepsilon}(x)dx\right)$ the space of L^p-integrable functions with respect to the measure $a_{i,\varepsilon}(x)dx$ and the corresponding norm is indicated by

$$|u|_{p,i,\varepsilon} := \left(\int_{\mathbb{R}^N} a_{i,\varepsilon}(x)|u|^p dx\right)^{\frac{1}{p}}, \quad p > 1.$$

Then we have the following result on the compact embedding.

Lemma 4.2. *For any $\varepsilon \in (0, s_1)$, the embedding $D_0^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*(s_i)}\left(\mathbb{R}^N, a_{i,\varepsilon}(x)dx\right)$ is compact.*

Proof. We refer to [13, Lemma 7.6] for the details. □

We note that for any compact set $\Omega \subset \mathbb{R}^N$ with $0 \notin \bar{\Omega}$, we have that $a_{i,\varepsilon}(x) \to a_{i,0}(x)$ uniformly for $i \in \{1, 2, \cdots, l\}$ and $x \in \Omega$ as $\varepsilon \to 0$. Now, for any $0 < \varepsilon < s_1$ fixed, let us consider the following problem:

$$\begin{cases} \Delta u + \displaystyle\sum_{i=1}^{l} \lambda_i a_{i,\varepsilon}(x) u^{2^*(s_i)-1} + u^{2^*-1} = 0 \text{ in } \mathbb{R}^N \backslash \{0\}, \\ u(x) > 0 \text{ in } \mathbb{R}^N \backslash \{0\}, \quad u \in D_0^{1,2}(\mathbb{R}^N), \end{cases} \tag{4.2}$$

whose energy functional is given by

$$\Phi_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dx - \sum_{i=1}^{\lambda_i} \frac{1}{2^*(s_i)} \int_{\mathbb{R}^N} a_{i,\varepsilon}(x)|u|^{2^*(s_i)}dx - \frac{1}{2^*}\int_{\mathbb{R}^N} |u|^{2^*} dx. \tag{4.3}$$

4.1 Nehari Manifold $\mathcal{N}_\varepsilon$

Consider the corresponding Nehari manifold

$$\mathcal{N}_\varepsilon := \Big\{ u \in D_0^{1,2}(\mathbb{R}^N)\backslash\{0\} \ : J_\varepsilon(u) = 0 \Big\}, \text{ where } J_\varepsilon(u) := \langle \Phi'_\varepsilon(u), u\rangle. \quad (4.4)$$

The following properties of $\mathcal{N}_\varepsilon$ are basic and the proofs are standard. For the reader's convenience, we give the details.

Lemma 4.3. *Let $N \geq 3, 0 = s_0 < s_1 < s_2 < \cdots < s_l < 2, \lambda_0 = 1$, there exists some $1 \leq k \leq l$ such that $\lambda_i > 0$ for $1 \leq i \leq k$ and $\lambda_i < 0$ for $k+1 \leq i \leq l$. Then for any $u \in D_0^{1,2}(\mathbb{R}^N)\backslash\{0\}$, there exists a unique $t_{(\varepsilon,u)} > 0$ such that $t_{(\varepsilon,u)}u \in \mathcal{N}_\varepsilon$. Further, $\mathcal{N}_\varepsilon$ is closed and bounded away from 0. Moreover, if $k \neq l$, for any fixed $u \in D_0^{1,2}(\mathbb{R}^N)\backslash\{0\}$, $t = t_{(\varepsilon,u)}$ is strictly increasing by ε in $[0, s_1)$.*

Proof. Firstly, we consider the case of that $k = l$. For any $0 \neq u \in D_0^{1,2}(\mathbb{R}^N)$, we set

$$b_{i,\varepsilon}(u) := \lambda_i \int_{\mathbb{R}^N} a_{i,\varepsilon}(x)|u|^{2^*(s_i)}dx > 0, \quad i = 1, 2, \cdots, l. \quad (4.5)$$

We note that $b_{i,\varepsilon}(u)$ is strictly decreasing by ε due to the monotonicity of $a_{i,\varepsilon}(x)$. Since

$$\Phi_\varepsilon(tu) = \frac{1}{2}\|u\|^2t^2 - \sum_{i=1}^{l} \frac{b_{i,\varepsilon}(u)}{2^*(s_i)}t^{2^*(s_i)} - \frac{1}{2^*}|u|_{2^*}^{2^*}t^{2^*}, \quad (4.6)$$

by a direct computation, we see that $\frac{d\Phi_\varepsilon(tu)}{dt} = 0$ has a unique solution $t_{(\varepsilon,u)} > 0$. Precisely, $t_{(\varepsilon,u)}$ is implicitly given by the following algebraic equation

$$\|u\|^2 - \sum_{i=1}^{l} b_{i,\varepsilon}(u)t^{2^*(s_i)-2} - |u|_{2^*}^{2^*}t^{2^*-2} = 0. \quad (4.7)$$

By Sobolev inequality, it is easy to see that there exists $\delta_\varepsilon > 0$ such that $t_{(\varepsilon,u)} \geq \delta_\varepsilon$ for any u satisfying $\|u\| = 1$. Hence, $\mathcal{N}_\varepsilon$ is bounded away form 0. Now, we prove that $t = t_{(\varepsilon,u)}$ is increasing by ε. Assume that $0 \leq \varepsilon_1 < \varepsilon_2 < s_1$, then we see that there exists a unique t_1 and t_2 such that

$$J_{\varepsilon_1}(t_1u) = J_{\varepsilon_2}(t_2u) = 0. \quad (4.8)$$

Recalling that $b_{i,\varepsilon}(u)$ is strictly decreasing by ε, we have that

$$J_{\varepsilon_2}(t_1u) > J_{\varepsilon_1}(t_1u) = 0 = J_{\varepsilon_2}(t_2u). \quad (4.9)$$

Noting that $J_{\varepsilon_2}(tu) \to -\infty$ as $t \to +\infty$, then by the uniqueness of $t_{\varepsilon_2,u}$, we see that $t_{(\varepsilon_2,u)} = t_2 > t_1$. Hence, $t_{(\varepsilon,u)}$ is strictly increasing by ε in $[0, s_1)$.

Secondly, we consider the case of that $k \neq l$. For the convenience, we denote $b_{0,\varepsilon}(u) \equiv |u|_{2^*}^{2^*}$. Then we see that

$$\Phi_\varepsilon(tu) = \frac{1}{2}\|u\|^2t^2 - \sum_{i=0}^{l} \frac{1}{2^*(s_i)}b_{i,\varepsilon}(u)t^{2^*(s_i)}. \quad (4.10)$$

For a given $u \neq 0$, we have

$$\frac{d}{dt}\Phi_\varepsilon(tu) := tf(t), \tag{4.11}$$

where

$$f(t) := \|u\|^2 - \sum_{i=0}^{l} b_{i,\varepsilon}(u)t^{2^*(s_i)-2}. \tag{4.12}$$

Noting that $\frac{d}{dt}\Phi_\varepsilon(tu) = 0$ with $t > 0$ if and only if $f(t) = 0$, hence the existence of $t_{(\varepsilon,u)}$ follows easily from the continuity of $f(t)$ and the facts that $f(0) = \|u\|^2 > 0$, $\lim\limits_{t\to+\infty} f(t) = -\infty$.

Now, we shall prove the uniqueness of $t_{(\varepsilon,u)}$. Set $\mathcal{A} := \{t > 0 \ : \ f(t) = 0\}$. Then, we see that $\mathcal{A} \neq \emptyset$. Let $t_0 := \inf\ \mathcal{A}$, then by it is easy to see that $t_0 > 0$ and $t_0 \in \mathcal{A}$, i.e., t_0 is the minimal positive root of $f(t) = 0$. Hence, we have $f(t) > 0$ for $0 < t < t_0$ and $f(t_0) = 0$. Next, we will show that $f'(t) < 0$ for $t > t_0$ and thus $f(t) < f(t_0) = 0$ for $t > t_0$. Indeed,

$$f'(t) = -\sum_{i=0}^{l}[2^*(s_i) - 2]b_{i,\varepsilon}(u)t^{2^*(s_i)-3}$$

$$:= -t^{2^*(s_{k+1})-3}\left[\sum_{i=0}^{l}[2^*(s_i) - 2]b_{i,\varepsilon}(u)t^{2^*(s_i)-2^*(s_{k+1})}\right]$$

and thus we only need to prove that

$$g(t) := \sum_{i=0}^{l}[2^*(s_i) - 2]b_{i,\varepsilon}(u)t^{2^*(s_i)-2^*(s_{k+1})} > 0 \text{ for } t > t_0. \tag{4.13}$$

By $f(t_0) = 0$, we have

$$b_{k+1,\varepsilon}(u) = \|u\|^2 t_0^{2-2^*(s_{k+1})} - \sum_{i\neq k+1} b_{i,\varepsilon}(u)t_0^{2^*(s_i)-2^*(s_{k+1})}. \tag{4.14}$$

Noting that $b_{i,\varepsilon}(u) > 0, 2^*(s_i) > 2^*(s_{k+1})$ for $0 \leq i \leq k$, then $b_{i,\varepsilon}(u)t^{2^*(s_i)-2^*(s_{k+1})}$ is increasing by t in $(t_0, +\infty)$. We also note that $b_{i,\varepsilon}(u) < 0, 2^*(s_i) < 2^*(s_{k+1})$ for $k+1 < i \leq l$. Hence, for this case, we also obtain that $b_{i,\varepsilon}(u)t^{2^*(s_i)-2^*(s_{k+1})}$ is increasing by t in $(t_0, +\infty)$. It is trivial that $2^*(s_i) - 2 > 0$ for $i = 0, 1, \cdots, l$.

Hence,

$$
\begin{aligned}
g(t) :=& \sum_{i\neq k+1} [2^*(s_i)-2] b_{i,\varepsilon}(u) t^{2^*(s_i)-2^*(s_{k+1})} + [2^*(s_{k+1})-2] b_{k+1,\varepsilon}(u) \\
>& \sum_{i\neq k+1} [2^*(s_i)-2] b_{i,\varepsilon}(u) t_0^{2^*(s_i)-2^*(s_{k+1})} + [2^*(s_{k+1})-2] b_{k+1,\varepsilon}(u) \\
=& \sum_{i\neq k+1} [2^*(s_i)-2] b_{i,\varepsilon}(u) t_0^{2^*(s_i)-2^*(s_{k+1})} \\
&+ [2^*(s_{k+1})-2]\left[\|u\|^2 t_0^{2-2^*(s_{k+1})} - \sum_{i\neq k+1} b_{i,\varepsilon}(u) t_0^{2^*(s_i)-2^*(s_{k+1})}\right] \\
=&[2^*(s_{k+1})-2]\|u\|^2 t_0^{2-2^*(s_{k+1})} + \\
&\sum_{i\neq k+1} [2^*(s_i)-2^*(s_{k+1})] b_{i,\varepsilon}(u) t_0^{2^*(s_i)-2^*(s_{k+1})}. \qquad (4.15)
\end{aligned}
$$

Noting $2^*(s_{k+1})-2>0$ and $[2^*(s_i)-2^*(s_{k+1})]b_{i,\varepsilon}(u)>0$ for $i\neq k+1$. Thus, (4.13) is proved and thereby we obtain the uniqueness of $t_{(\varepsilon,u)}$. In particular, for the case of $k\neq l$, we can not obtain the monotonicity of $t_{(\varepsilon,u)}$ by ε in $[0,s_1)$. $\square$

Lemma 4.4. *Under the assumption of Lemma 4.3, any $(PS)_c$ sequence of $\Phi_\varepsilon(u)$ is bounded in $D_0^{1,2}(\mathbb{R}^N)$.*

Proof. Since $\{u_n\}$ is a $(PS)_c$ sequence, i.e., $\Phi_\varepsilon(u_n)=c+o(1)$ and $\langle \Phi'_\varepsilon(u_n),u_n\rangle = o(1)\|u_n\|$, we have

$$
\begin{aligned}
&c+o(1)+o(1)\|u_n\| \\
=&\left[\frac{1}{2}-\frac{1}{2^*(s_k)}\right]\|u_n\|^2 + \sum_{i=1}^{l}\left[\frac{1}{2^*(s_k)}-\frac{1}{2^*(s_i)}\right] b_{i,\varepsilon}(u_n) + \left[\frac{1}{2^*(s_k)}-\frac{1}{2^*}\right]|u_n|_{2^*}^{2^*} \\
>&\left[\frac{1}{2}-\frac{1}{2^*(s_k)}\right]\|u_n\|^2, \qquad (4.16)
\end{aligned}
$$

which implies that $\{u_n\}$ is bounded in $D_0^{1,2}(\mathbb{R}^N)$. $\square$

Define

$$c_\varepsilon := \inf_{u\in\mathcal{N}_\varepsilon} \Phi_\varepsilon(u) \qquad (4.17)$$

and

$$\delta_\varepsilon := \inf_{u\in\mathcal{N}_\varepsilon} \|u\|. \qquad (4.18)$$

Lemma 4.5. *Under the assumptions of Lemma 4.3 and furthermore we suppose that $k=l$, i.e., all λ_is are positive. Then δ_ε is strictly increasing by $\varepsilon\in[0,s_1)$, i.e., $0<\delta_0\le\delta_{\varepsilon_1}<\delta_{\varepsilon_2}$ provided $0\le\varepsilon_1<\varepsilon_2<s_1$.*

Proof. It follows from the strictly monotonicity of $t_{(\varepsilon,u)}$ in Lemma 4.3. $\square$

Lemma 4.6. *Under the assumptions of Lemma 4.3. Let $\{u_n\}$ be a $(PS)_c$ sequence of $\Phi_\varepsilon\big|_{\mathcal{N}_\varepsilon}$ i.e.,*

$$\begin{cases} \Phi_\varepsilon(u_n) \to c \\ \Phi'_\varepsilon\big|_{\mathcal{N}_\varepsilon}(u_n) \to 0 \text{ in the dual space of } D_0^{1,2}(\mathbb{R}^N) \end{cases},$$

then $\{u_n\}$ is also a $(PS)_c$ sequence of Φ_ε.

Proof. For any $u \in \mathcal{N}_\varepsilon$, we have

$$J_\varepsilon(u) = \|u\|^2 - \sum_{i=1}^{l} b_{i,\varepsilon}(u) - |u|_{2^*}^{2^*} = 0. \tag{4.19}$$

Consider the case of $k = l$, we have

$$\begin{aligned} \langle J'_\varepsilon(u), u\rangle =& 2\|u\|^2 - \sum_{i=1}^{l} 2^*(s_i) b_{i,\varepsilon}(u) - 2^*|u|_{2^*}^{2^*} \\ =& -\Big[\big[\sum_{i=1}^{l}\big(2^*(s_i)-2\big)b_{i,\varepsilon}(u)\big] + (2^*-2)|u|_{2^*}^{2^*}\Big] \\ \leq& -\big[2^*(s_l)-2\big]\big[\sum_{i=1}^{l} b_{i,\varepsilon}(u) + |u|_{2^*}^{2^*}\big] \\ =& -\big[2^*(s_l)-2\big]\|u\|^2 \\ \leq& -\big[2^*(s_l)-2\big]\delta_\varepsilon^2 < 0. \end{aligned} \tag{4.20}$$

However, when $k \neq l$, we note that $b_{i,\varepsilon}(u) > 0$ for $1 \leq i \leq k$ and $b_{i,\varepsilon}(u) > 0$ for $k+1 \leq i \leq l$. Here we view $b_{0,\varepsilon}(u)$ as $|u|_{2^*}^{2^*}$. Hence, we have

$$\begin{aligned} \langle J'_\varepsilon(u), u\rangle =& 2\|u\|^2 - \sum_{i=0}^{l} 2^*(s_i) b_{i,\varepsilon}(u) \\ <& 2\|u\|^2 - 2^*(s_k)\sum_{i=0}^{l} b_{i,\varepsilon}(u) \\ =& -\left[2^*(s_k)-2\right]\|u\|^2 \leq -\left[2^*(s_k)-2\right]\delta_\varepsilon^2 < 0. \end{aligned} \tag{4.21}$$

Apply the similar arguments as Lemma 4.4, we see that $\{u_n\}$ is bounded in $D_0^{1,2}(\mathbb{R}^N)$. Let $\{t_n\} \subset \mathbb{R}$ be a sequence of multipliers satisfying

$$\Phi'_\varepsilon(u_n) = \Phi'_\varepsilon\big|_{\mathcal{N}_\varepsilon}(u_n) + t_n J'_\varepsilon(u_n). \tag{4.22}$$

Testing by u_n, we obtain that

$$t_n\langle J'_\varepsilon(u_n), u_n\rangle \to 0. \tag{4.23}$$

By (4.20) or (4.21) and (4.23), we see that

$$t_n \to 0 \text{ as } n \to +\infty. \tag{4.24}$$

Noting that $J'_\varepsilon(u_n)$ is bounded due to the boundedness of $\{u_n\}$, hence by (4.22) and (4.24), we have $\Phi'_\varepsilon(u_n) \to 0$ as $n \to +\infty$. $\square$

Remark 4.1. *By the formula* (4.16)*, for $\varepsilon > 0$, we have that*

$$c_\varepsilon \geq [\frac{1}{2} - \frac{1}{2^*(s_k)}]\delta_\varepsilon^2 > 0. \tag{4.25}$$

Especially, when $k = l$, by Lemma 4.5, we have

$$c_\varepsilon > [\frac{1}{2} - \frac{1}{2^*(s_l)}]\delta_0^2 > 0. \tag{4.26}$$

For the case of $k \neq l$, we will prove that c_ε is also achieved by some u_ε and u_ε is a mountain pass type solution (see Theorem 4.1). Set

$$\tilde{\Phi}_\varepsilon(u) := \frac{1}{2}\|u\|^2 - \sum_{i=1}^{k} \int_{\mathbb{R}^N} \frac{\lambda_i}{2^*(s_i)} a_{i,\varepsilon}(x)|u|^{2^*(s_i)}. \tag{4.27}$$

It follows that there exists some $\tilde{\delta}_0 > 0$ such that

$$\inf_{u \in D_0^{1,2}(\mathbb{R}^N)\setminus\{0\}} \max_{t>0} \tilde{\Phi}_\varepsilon(tu) \geq [\frac{1}{2} - \frac{1}{2^*(s_k)}]\tilde{\delta}_0^2 > 0. \tag{4.28}$$

Then it follows that

$$c_\varepsilon = \Phi_\varepsilon(u_\varepsilon) = \max_{t>0} \Phi_\varepsilon(tu_\varepsilon) > \max_{t>0} \tilde{\Phi}_\varepsilon(tu_\varepsilon) \geq \tilde{c}_\varepsilon \geq [\frac{1}{2} - \frac{1}{2^*(s_k)}]\tilde{\delta}_0^2 > 0. \tag{4.29}$$

Lemma 4.7. *If $k = l$, c_ε is strictly increasing by ε in $[0, s_1)$.*

Proof. Let $0 \neq u \in D_0^{1,2}(\mathbb{R}^N)$ be fixed. For any $\varepsilon \in [0, s_1)$, by Lemma 4.3, there exists $t_\varepsilon > 0$ such that $t_\varepsilon u \in \mathcal{N}_\varepsilon$ and t_ε is implicity given by

$$\|u\|^2 - \sum_{i=1}^{l} b_{i,\varepsilon}(u) t_\varepsilon^{2^*(s_i)-2} - |u|_{2^*}^{2^*} t_\varepsilon^{2^*-2} = 0 \tag{4.30}$$

By the Implicit Function Theorem, we see that $t(\varepsilon) \in C^1(\mathbb{R})$ and $\frac{d}{d\varepsilon}t(\varepsilon) > 0$ due to Lemma 4.3. Hence, recalling that $b_{i,\varepsilon}(u)$ is strictly decreasing by ε and

the formula (4.30), we have

$$
\begin{aligned}
&\frac{d}{d\varepsilon}\Phi_\varepsilon(t_\varepsilon u)\\
=&\frac{d}{d\varepsilon}\Big[\frac{1}{2}\|u\|^2 t_\varepsilon^2-\sum_{i=1}^{l}\frac{b_{i,\varepsilon}(u)}{2^*(s_i)}t_\varepsilon^{2^*(s_i)}-\frac{1}{2^*}|u|_{2^*}^{2^*}t_\varepsilon^{2^*}\Big]\\
=&\frac{t_\varepsilon'}{t_\varepsilon}\big[\|u\|^2 t_\varepsilon^2-\sum_{i=1}^{l}b_{i,\varepsilon}(u)t_\varepsilon^{2^*(s_i)}-|u|_{2^*}^{2^*}t_\varepsilon^{2^*}\big]-\sum_{i=1}^{l}\frac{b_{i,\varepsilon}'(u)}{2^*(s_i)}t_\varepsilon^{2^*(s_i)}\\
=&-\sum_{i=1}^{l}\frac{b_{i,\varepsilon}'(u)}{2^*(s_i)}t_\varepsilon^{2^*(s_i)}\\
>&0,
\end{aligned}
\tag{4.31}
$$

therefore, c_ε is strictly increasing by ε in $[0, s_1)$. $\square$

4.2 Existence of positive ground state of the approximating problem (4.2)

In this subsection, we assume that $\varepsilon \in (0, s_1)$ is fixed.

Theorem 4.1. *Let $N \geq 3, 0 < s_1 < s_2 < \cdots < s_l < 2$. Suppose that there exists some $1 \leq k \leq l$ such that $\lambda_i > 0$ for $1 \leq i \leq k$ and $\lambda_i < 0$ for $k+1 \leq i \leq l$. Furthermore, if $N = 3$ and $k \neq l$, we assume that either $s_1 < 1$ or $1 \leq s_1 < 2$ with $\max\{|\lambda_{k+1}|, \cdots, |\lambda_l|\}$ small enough. Then for any $\varepsilon \in (0, s_1)$, problem* (4.2) *possesses a positive ground state solution having the least energy*

$$c_\varepsilon < \frac{1}{N}S^{\frac{N}{2}}. \tag{4.32}$$

In particular, if $k = l$, c_ε is increasing strictly by ε.

We postpone the proof of Theorem 4.1 and do a little preparation before that. Denote

$$\Psi(u) := \frac{1}{2}\|u\|^2 - \frac{1}{2^*}|u|_{2^*}^{2^*} \quad u \in D_0^{1,2}(\mathbb{R}^N). \tag{4.33}$$

It is well known that

$$\min_{u\in D_0^{1,2}(\mathbb{R}^N)\backslash\{0\}}\max_{t>0}\Psi(tu) = \frac{1}{N}S^{\frac{N}{2}}. \tag{4.34}$$

By Lemma 4.4, any $(PS)_c$ sequence of Φ_ε is bounded in $D_0^{1,2}(\mathbb{R}^N)$. Hence, we may give the following proposition:

Proposition 4.1. *Let $N \geq 3, 0 < s_1 < s_2 < \cdots < s_l < 2$, there exists some $1 \leq k \leq l$ such that $\lambda_i > 0$ for $1 \leq i \leq k$ and $\lambda_i < 0$ for $k+1 \leq i \leq l$. Take $\varepsilon \in (0, s_1)$ and assume that $\{u_n\}$ is a $(PS)_c$ sequence of Φ_ε, i.e.,*

$$\begin{cases}\Phi_\varepsilon(u_n) \to c,\\ \Phi_\varepsilon'(u_n) \to 0.\end{cases} \tag{4.35}$$

Up to a subsequence, we may assume that $u_n \rightharpoonup u_0$ in $D_0^{1,2}(\mathbb{R}^N)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^N$. Denote $\tilde{u}_n := u_n - u_0$, then $\{\tilde{u}_n\}$ is a PS sequence of Ψ.

Proof. Basing on the Lemma 4.1, by Hölder inequality and Hardy Sobolev inequality, it is easy to prove that

$$\int_{\mathbb{R}^N} a_{i,\varepsilon}(x)|u_n|^{2^*(s_i)-1}hdx - \int_{\mathbb{R}^N} a_{i,\varepsilon}(x)|u_0|^{2^*(s_i)-1}hdx = o(1)\|h\|. \tag{4.36}$$

Since $\{u_n\}$ is a $(PS)_c$ sequence of Φ_ε, we see that $\Phi'_\varepsilon(u_0) = 0$. Then it follows that

$$\int_{\mathbb{R}^N} \nabla(u_n - u_0)\nabla hdx - \int_{\mathbb{R}^N} \left[|u_n|^{2^*-2}u_n - |u_0|^{2^*-2}u_0\right]hdx = o(1)\|h\|. \tag{4.37}$$

By the Brézis-Lieb Lemma, we see that

$$|u_n|^{2^*-2}u_n - |u_0|^{2^*-2}u_0 - |u_n - u_0|^{2^*-2}(u_n - u_0) \to 0 \text{ strongly in } L^{\frac{2^*}{2^*-1}}(\mathbb{R}^N). \tag{4.38}$$

Hence, by (4.37) and (4.38), this proposition if proved. □

Corollary 4.1. *Let $N \geq 3, 0 < s_1 < s_2 < \cdots < s_l < 2$, there exists some $1 \leq k \leq l$ such that $\lambda_i > 0$ for $1 \leq i \leq k$ and $\lambda_i < 0$ for $k+1 \leq i \leq l$. Then for any $\varepsilon \in (0, s_1)$, Φ_ε satisfies $(PS)_c$ condition if $c < \frac{1}{N}S^{\frac{N}{2}}$.*

Proof. Let $\{u_n\}$ be a $(PS)_c$ sequence of Φ_ε with $c < \frac{1}{N}S^{\frac{N}{2}}$. Up to a subsequence, we assume that $u_n \rightharpoonup u_0$ in $D_0^{1,2}(\mathbb{R}^N)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^N$. We prove this corollary by the way of negation. If $u_n \not\to u_0$ in $D_0^{1,2}(\mathbb{R}^N)$, then by Proposition 4.1, we see that $\tilde{u_n} := u_n - u_0$ is a PS sequence of Ψ and $\tilde{u}_n \not\to 0$. Then it is easy to prove that

$$\lim_{n\to+\infty} \Psi(\tilde{u}_n) \geq \frac{1}{N}S^{\frac{N}{2}}. \tag{4.39}$$

By Brézis-Lieb Lemma again, we have that

$$\Phi_\varepsilon(u_n) = \Phi_\varepsilon(u_0) + \Psi(\tilde{u}_n) + o(1). \tag{4.40}$$

Since $\Phi'_\varepsilon(u_0) = 0$, it is easy to see that $\Phi_\varepsilon(u_0) \geq 0$. Hence, by (4.39) and (4.40), we obtain that

$$c = \lim_{n\to+\infty} \Phi_\varepsilon(u_n) \geq \frac{1}{N}S^{\frac{N}{2}}, \tag{4.41}$$

a contradiction. Hence, $u_n \to u_0$ strongly in $D_0^{1,2}(\mathbb{R}^N)$ and it follows that $\Phi_\varepsilon(u_0) = c$. □

Lemma 4.8. *Let $N \geq 3, 0 < s_1 < s_2 < \cdots < s_l < 2$. Suppose that there exists some $1 \leq k \leq l$ such that $\lambda_i > 0$ for $1 \leq i \leq k$ and $\lambda_i < 0$ for $k+1 \leq i \leq l$. Furthermore, if $N = 3$ and $k \neq l$, we assume that either $s_1 < 1$ or $1 \leq s_1 < 2$ with $\max\{|\lambda_{k+1}|, \cdots, |\lambda_l|\}$ small enough. Take $\varepsilon \in [0, s_1)$ and suppose that c_ε is given by (4.17), then we have*

$$c_\varepsilon < \frac{1}{N}S^{\frac{N}{2}}. \tag{4.42}$$

Proof. Let $U(x) := \dfrac{[N(N-2)]^{\frac{N-2}{4}}}{[1+|x|^2]^{\frac{N-2}{2}}}$. For the case of $k = l$, it is easy to see that

$$c_\varepsilon \le \max_{t>0} \Phi_\varepsilon(tU) < \max_{t>0} \Psi(tU) = \frac{1}{N} S^{\frac{N}{2}}. \tag{4.43}$$

Next, we consider the case of $k \neq l$. When $1 \le s_1 < 2$ with $\max\{|\lambda_{k+1}|, \cdots, |\lambda_l|\}$ small enough, a direct computation shows that (4.43) is also satisfied. And we note that the small bound can be chosen independent of ε for ε small enough. When $0 < s_1 < 1$ and $k \neq l$, we let $0 \neq x_0 \in \mathbb{R}^N$, and $\psi \in D_0^{1,2}(\mathbb{R}^N)$ be a nonnegative function such that $\psi \equiv 1$ on $B(0,\rho), 0 < \rho < |x_0|$. For $\sigma > 0$, we define

$$U_\sigma(x) := \sigma^{\frac{2-N}{2}} U\left(\frac{x - x_0}{\sigma}\right), u_\sigma(x) := \psi(x - x_0) U_\sigma(x).$$

Noting that $\varepsilon > 0$ is fixed, a direct computation shows that

$$\int_{\mathbb{R}^N} a_{i,\varepsilon}(x) |u_\sigma(x)|^{2^*(s_i)} dx = O\left(\sigma^{s_i}\right), \ i = 1, 2, \cdots, l, \tag{4.44}$$

and

$$\int_{\mathbb{R}^N} |\nabla u_\sigma(x)|^2 dx = S^{\frac{N}{2}} + O(\sigma^{N-2}); \int_{\mathbb{R}^N} |u_\sigma(x)|^{2^*} dx = S^{\frac{N}{2}} + O(\sigma^N). \tag{4.45}$$

Since $\lambda_1 > 0$ and $s_1 < \min\{s_2, \cdots, s_l, N-2\}$ under the assumptions, then it is easy to see that

$$\max_{t>0} \Phi_\varepsilon(tu_\sigma) < \frac{1}{N} S^{\frac{N}{2}} \text{ for } \sigma \text{ small enough.} \tag{4.46}$$

Hence, by the definition of c_ε, we obtain that $c_\varepsilon < \dfrac{1}{N} S^{\frac{N}{2}}$. $\square$

Proof of Theorem 4.1: Let $\{u_n\} \subset \mathcal{N}_\varepsilon$ be a minimizing sequence of c_ε. Then by Lemma 4.6, we see that $\{u_n\}$ is also a $(PS)_{c_\varepsilon}$ sequence of Φ_ε. Under the assumptions, by Lemma 4.8, we have $c_\varepsilon < \dfrac{1}{N} S^{\frac{N}{2}}$. By Corollary 4.1, we observe that Φ_ε satisfies the $(PS)_{c_\varepsilon}$ condition. Up to a subsequence, we may assume that $u_n \to u_0$ strongly in $D_0^{1,2}(\mathbb{R}^N)$ and $\Phi_\varepsilon(u_0) = c_\varepsilon$. Hence, u_0 is a minimizer of c_ε. Noting that Φ_ε is even, we see that $|u_0|$ is also a minimizer of c_ε. Hence, without loss of generality, we may assume that $u_0 \ge 0$. Then, we see that $\Phi'_\varepsilon(u_0) = 0$. By the maximum principle, we have that $u_0 > 0$ in $\mathbb{R}^N \backslash \{0\}$. Hence, u_0 is a positive ground state solution of problem (4.2) and Theorem 4.1 is proved. $\square$

Remark 4.2. *For $\varepsilon \in [0, s_1)$, we define the mountain pass value*

$$\tilde{c}_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{t \in [0,1]} \Phi_\varepsilon(\gamma(t)), \tag{4.47}$$

where $\Gamma_\varepsilon := \left\{\gamma(t) \in C\left([0,1], D_0^{1,2}(\mathbb{R}^N)\right) : \gamma(0) = 0, \Phi_\varepsilon(\gamma(1)) < 0\right\}$. *It is standard to prove that* $c_\varepsilon = \tilde{c}_\varepsilon$ *and any ground state solution of* (4.2) *is a mountain pass solution provided that* $\varepsilon > 0$. *Precisely, by the definition and the result of Lemma 4.3, it is easy to see that* $\tilde{c}_\varepsilon \le c_\varepsilon$ *for all* $\varepsilon \in [0, s_1)$. *When* $\varepsilon > 0$, *by Corollary 4.1,* Φ_ε *satisfies* $(PS)_{\tilde{c}_\varepsilon}$ *condition. Hence, there exists a mountain pass solution* $\tilde{u}_\varepsilon$ *such that* $\Phi_\varepsilon(\tilde{u}_\varepsilon) = \tilde{c}_\varepsilon$. *It follows that*

$$\tilde{c}_\varepsilon = \Phi_\varepsilon(\tilde{u}_\varepsilon) = \max_{t>0} \Phi_\varepsilon(t\tilde{u}_\varepsilon) \ge \min_{u\neq 0} \max_{t>0} \Phi_\varepsilon(tu) = c_\varepsilon, \tag{4.48}$$

thus we can obtain the reverse inequality. Hence, we have $\tilde{c}_\varepsilon = c_\varepsilon$ *for* $\varepsilon \in (0, s_1)$.

Lemma 4.9. *Under the assumptions of Theorem 1.1, we have that* $\limsup_{\varepsilon\to 0^+} \tilde{c}_\varepsilon \le \tilde{c}_0$. *Moreover, if* $k = l$, *we have that* $\tilde{c}_\varepsilon > \tilde{c}_0$ *and thus* $\lim_{\varepsilon\to 0^+} \tilde{c}_\varepsilon = \tilde{c}_0$.

Proof. For any $\delta > 0$, there exists $\gamma_0 \in \Gamma_0$ such that

$$\max_{t\in[0,1]} \Phi_0(\gamma_0(t)) < \tilde{c}_0 + \delta. \tag{4.49}$$

Denote $\gamma_0(1) = \phi$, since $\gamma_0 \in \Gamma_0$, we have $\Phi_0(\phi) < 0$. By the Lebesgue's dominated convergence theorem, we have

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} \lambda_i a_{i,\varepsilon}(x)|\phi|^{2^*(s_i)}dx = \int_{\mathbb{R}^N} \lambda_i \frac{1}{|x|^{s_i}}|\phi|^{2^*(s_i)}dx. \tag{4.50}$$

Hence, by the continuity, we see that $\gamma_0 \in \Gamma_\varepsilon$ when ε is small enough. Now, take $\varepsilon_n \downarrow 0$ and denote $t_n \in [0,1]$ such that

$$\Phi_{\varepsilon_n}(\gamma_0(t_n)) = \max_{t\in[0,1]} \Phi_{\varepsilon_n}(\gamma_0(t)). \tag{4.51}$$

Up to a subsequence if necessary, we may assume that $t_n \to t^* \in [0,1]$. Set $u_n := \gamma_0(t_n)$ and $u^* := \gamma_0(t^*)$, since $\gamma_0 \in C([0,1], D_0^{1,2}(\mathbb{R}^N))$, we obtain that $u_n \to u^*$ strongly in $D_0^{1,2}(\mathbb{R}^N)$. Hence, we have

$$\Phi_{\varepsilon_n}(u_n) = \Phi_{\varepsilon_n}(u^*) + o(1). \tag{4.52}$$

On the other hand, by the Lebesgue's dominated convergence theorem again, we have

$$\Phi_{\varepsilon_n}(u^*) = \Phi_0(u^*) + o(1). \tag{4.53}$$

Then by (4.52) and (4.53), we have

$$\begin{aligned}\tilde{c}_\varepsilon \le& \Phi_{\varepsilon_n}(\gamma_0(t_n)) = \Phi_{\varepsilon_n}(u_n)\\ =& \Phi_0(u^*) + o(1) \le \max_{t\in[0,1]} \Phi_0(\gamma_0(t)) + o(1)\\ \le& \tilde{c}_0 + \delta + o(1).\end{aligned} \tag{4.54}$$

Hence, $\limsup_{n\to+\infty} \tilde{c}_{\varepsilon_n} \le \tilde{c}_0$ due to the arbitrariness of δ.

Moreover, if $k = l$, noting that $a_{i,\varepsilon}(x)$ is decreasing by ε for all $i = 1, 2, \cdots, l$, it is easy to see that $\tilde{c}_\varepsilon \geq \tilde{c}_0$. By Theorem 4.1 and Remark 4.2, $\tilde{c}_\varepsilon = c_\varepsilon$ can be achieved. Hence, $\tilde{c}_\varepsilon > \tilde{c}_0$. It is also trivial that $\lim\limits_{\varepsilon\to0^+} \tilde{c}_\varepsilon \geq \tilde{c}_0$. Hence,

$$\lim_{\varepsilon\to0^+} \tilde{c}_\varepsilon = \tilde{c}_0. \tag{4.55}$$

□

Lemma 4.10. *$\tilde{c}_0 \leq c_0$ and thus $\limsup\limits_{\varepsilon\to0^+} c_\varepsilon \leq c_0$. Especially, $\tilde{c}_0 = c_0$ and $\lim\limits_{\varepsilon\to0^+} c_\varepsilon = c_0$ if $k = l$.*

Proof. For any $0 \neq u \in D_0^{1,2}(\mathbb{R}^N)$, since $\gamma(t) := tTu \in \Gamma_0(t)$ for T large enough, then by the definition of c_0 and $\tilde{c}_0$, it is easy to see that

$$\tilde{c}_0 \leq c_0. \tag{4.56}$$

On the other hand, by Remark 4.2 and Lemma 4.9, we have

$$\tilde{c}_0 \geq \limsup_{\varepsilon\to0^+} \tilde{c}_\varepsilon = \limsup_{\varepsilon\to0^+} c_\varepsilon. \tag{4.57}$$

Moreover, if $k = l$, by Lemma 4.7, $c_\varepsilon > c_0$ for any $\varepsilon > 0$, combining with Lemma 4.9, we obtain the reverse inequality

$$\tilde{c}_0 = \lim_{\varepsilon\to0^+} \tilde{c}_\varepsilon = \lim_{\varepsilon\to0^+} c_\varepsilon \geq c_0. \tag{4.58}$$

Hence, by (4.56) and (4.58), we see that $c_0 = \tilde{c}_0$. □

Remark 4.3. *When $\varepsilon = 0$, since it is not trivial to see that c_0 is a ground state value, we can not obtain that $\tilde{c}_0 = c_0$ by the arguments as the case of $\varepsilon > 0$ that mentioned in Remark 4.2. However, if $k = l$, by Lemma 4.10 above, we still obtain that $\tilde{c}_0 = c_0$. For the case of $k \neq l$, since we can not obtain the monotonicity of c_ε, we are unable to get the conclusion of $\tilde{c}_0 = c_0$ up to now. However, we note that after the results established in present paper, we will see that this relationship still holds. Especially, c_0 can be attained. We can also obtain that $\lim\limits_{\varepsilon\to0^+} c_\varepsilon = c_0$ for the case $k \neq l$.*

5 Interpolation Inequalities and Pohozaev Identity

The following Propositions 5.1-5.2 are proved in [13] and Proposition 5.3 is obtained in [9]. Define

$$\vartheta(s_1, s_2) := \frac{N(s_2 - s_1)}{s_2(N - s_1)} \quad \text{for } \ 0 \leq s_1 \leq s_2 \leq 2. \tag{5.1}$$

Proposition 5.1. *(see [13, Corollary 2.1]) Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be an open set. Assume $0 \leq s_1 < 2$. Then for any $s_2 \in [s_1, 2]$ and $\theta \in [\vartheta(s_1, s_2), 1]$, there exists $C(\theta) > 0$ such that*

$$|u|_{2^*(s_1),s_1} \leq C(\theta)\|u\|^\theta |u|^{1-\theta}_{2^*(s_2),s_2} \tag{5.2}$$

for all $u \in D^{1,2}_0(\Omega)$, where $\|u\| := \big(\int_\Omega |\nabla u|^2 dx\big)^{\frac{1}{2}}$.

Define

$$\varsigma(s_1, s_2) := \frac{(N - s_1)(2 - s_2)}{(N - s_2)(2 - s_1)} \quad \text{for } \ 0 \leq s_1 \leq s_2 \leq 2. \tag{5.3}$$

Proposition 5.2. *(see [13, Corollary 2.2]) Let $\Omega \subset \mathbb{R}^N (N \geq 3)$ be an open set. Assume $0 < s_2 \leq 2$. Then for any $s_1 \in [0, s_2]$ and $\sigma \in [0, \varsigma(s_1, s_2)]$, there exists a $C(\sigma) > 0$ such that*

$$|u|_{2^*(s_2),s_2} \leq C(\sigma)\|u\|^{1-\sigma} |u|^{\sigma}_{2^*(s_1),s_1} \tag{5.4}$$

for all $u \in D^{1,2}_0(\Omega)$

Proposition 5.3. *(see [9, Proposition 2.1]) Let $u \in H^1(\Omega)\backslash\{0\}$ be a solution to the equation $-\Delta u = g(x, u)$ and $G(x, u) = \int_0^u g(x, s)ds$ is such that $G\big(\cdot, u(\cdot)\big)$ and $x_i G_{x_i}\big(\cdot, u(\cdot)\big)$ are in $L^1(\Omega)$, then u satisfies:*

$$\int_{\partial\Omega} |\nabla u|^2 x \cdot \eta dS_x = 2N \int_\Omega G(x, u)dx + 2\sum_{i=1}^N \int_\Omega x_i G_{x_i}(x, u)dx - (N-2)\int_\Omega |\nabla u|^2 dx,$$

where Ω is a regular domain in $\mathbb{R}^N$ and η denotes the unitary exterior normal vector to $\partial\Omega$. Moreover, if $\Omega = \mathbb{R}^N$, then

$$2N \int_{\mathbb{R}^N} G(x, u)dx + 2\sum_{i=1}^N \int_{\mathbb{R}^N} x_i G_{x_i}(x, u)dx = (N - 2)\int_{\mathbb{R}^N} |\nabla u|^2 dx.$$

Corollary 5.1. *For $\varepsilon > 0$ small enough, we still have that for any $\theta \in [\vartheta(s_i, s_j), 1]$ if $0 < s_i \leq s_j < 2$,*

$$\left(\int_{\mathbb{R}^N} a_{i,\varepsilon}(x)|u|^{2^*(s_i)} dx\right)^{\frac{1}{2^*(s_i)}} \leq C(\theta)\|u\|^\theta \left(\int_{\mathbb{R}^N} a_{j,\varepsilon}(x)|u|^{2^*(s_j)} dx\right)^{\frac{1-\theta}{2^*(s_j)}} \tag{5.5}$$

And for any $\sigma \in [0, \varsigma(s_i, s_j)]$ if $0 < s_i \leq s_j < 2$,

$$\left(\int_{\mathbb{R}^N} a_{j,\varepsilon}(x)|u|^{2^*(s_j)} dx\right)^{\frac{1}{2^*(s_j)}} \leq C(\sigma)\|u\|^{1-\sigma} \left(\int_{\mathbb{R}^N} a_{i,\varepsilon}(x)|u|^{2^*(s_i)} dx\right)^{\frac{\sigma}{2^*(s_i)}}. \tag{5.6}$$

Proof. We replace dx by the new measure $d\nu := \begin{cases} \frac{dx}{|x|^{-\varepsilon}} \text{ if } |x| \le 1 \\ \frac{dx}{|x|^{\varepsilon}} \text{ if } |x| > 1 \end{cases}$. Recalling the embedding relationship in Lemma 4.1, by the same arguments as the the proofs of [13, Corollary 2.1 and Corollary 2.2], we can obtain the results of (5.5) and (6.10). We omit the details. $\square$

Corollary 5.2. *Let $N \ge 3, 0 < s_i < 2$ and $\varepsilon \in (0, s_1)$. Then any solution of* (4.2) *satisfies*

$$\int_{\mathbb{B}_1} \sum_{i=1}^{l} \frac{\lambda_i a_{i,\varepsilon}(x)}{2^*(s_i)} |u|^{2^*(s_i)} dx = \int_{\mathbb{B}_1^c} \sum_{i=1}^{l} \frac{\lambda_i a_{i,\varepsilon}(x)}{2^*(s_i)} |u|^{2^*(s_i)} dx. \tag{5.7}$$

Proof. Take $G(x,u) = \sum_{i=1}^{l} \frac{1}{2^*(s_i)} \lambda_i a_{i,\varepsilon}(x) |u|^{2^*(s_i)} + \frac{1}{2^*} u^{2^*}$. By Proposition 5.3, we have

$$\begin{aligned} &2N \int_{\mathbb{R}^N} \Big[\sum_{i=1}^{l} \frac{\lambda_i}{2^*(s_i)} a_{i,\varepsilon}(x) |u|^{2^*(s_i)} + \frac{1}{2^*} |u|^{2^*} \Big] dx \\ &+ 2 \sum_{j=1}^{N} \int_{\mathbb{R}^N} \sum_{i=1}^{l} \frac{\lambda_i}{2^*(s_i)} \frac{\partial}{\partial x_j} a_{i,\varepsilon}(x) |u|^{2^*(s_i)} x_j \\ =&(N-2) \int_{\mathbb{R}^N} |\nabla u|^2 dx. \end{aligned} \tag{5.8}$$

Noting that

$$\frac{\partial}{\partial x_j} a_{i,\varepsilon}(x) = \begin{cases} -(s_i - \varepsilon) \frac{1}{|x|^{s_i + 2 - \varepsilon}} x_j & \text{for } |x| < 1, \\ -(s_i + \varepsilon) \frac{1}{|x|^{s_i + 2 + \varepsilon}} x_j & \text{for } |x| > 1, \end{cases} \tag{5.9}$$

we obtain that

$$\sum_{j=1}^{N} \frac{\partial}{\partial x_j} a_{i,\varepsilon}(x) x_j = \begin{cases} -(s_i - \varepsilon) a_{i,\varepsilon}(x), & |x| < 1, \\ -(s_i + \varepsilon) a_{i,\varepsilon}(x), & |x| > 1. \end{cases} \tag{5.10}$$

Then, substitute into (5.8), we obtain that

$$\begin{aligned}
&2N\int_{\mathbb{R}^N}\Big[\sum_{i=1}^{l}\frac{\lambda_i}{2^*(s_i)}a_{i,\varepsilon}(x)|u|^{2^*(s_i)}+\frac{1}{2^*}|u|^{2^*}\Big]dx\\
&-\int_{\mathbb{R}^N}\Big[\sum_{i=1}^{l}\frac{2s_i\lambda_i}{2^*(s_i)}a_{i,\varepsilon}(x)|u|^{2^*(s_i)}\Big]dx\\
&+2\varepsilon\int_{\mathbb{B}_1}\Big[\sum_{i=1}^{l}\frac{\lambda_i a_{i,\varepsilon}(x)}{2^*(s_i)}|u|^{2^*(s_i)}\Big]dx\\
&-2\varepsilon\int_{\mathbb{B}_1^c}\Big[\sum_{i=1}^{l}\frac{\lambda_i a_{i,\varepsilon}(x)}{2^*(s_i)}|u|^{2^*(s_i)}\Big]dx\\
=&(N-2)\int_{\mathbb{R}^N}|\nabla u|^2dx.
\end{aligned}\tag{5.11}$$

On the other hand, since u is a solution, we have

$$\int_{\mathbb{R}^N}|\nabla u|^2dx=\int_{\mathbb{R}^N}\Big[\sum_{i=1}^{l}\lambda_i a_{i,\varepsilon}(x)|u|^{2^*(s_i)}+|u|^{2^*}\Big]dx.\tag{5.12}$$

Hence, by (5.11) and (5.12), we get

$$\int_{\mathbb{B}_1}\Big[\sum_{i=1}^{l}\frac{\lambda_i a_{i,\varepsilon}(x)}{2^*(s_i)}|u|^{2^*(s_i)}\Big]dx=\int_{\mathbb{B}_1^c}\Big[\sum_{i=1}^{l}\frac{\lambda_i a_{i,\varepsilon}(x)}{2^*(s_i)}|u|^{2^*(s_i)}\Big]dx.\tag{5.13}$$

□

6 Proof of Theorem 1.1

6.1 Preliminary

Remark 6.1. *For $\forall\ \varepsilon\in(0,s_1)$, by Theorem 4.1, problem (4.2) possesses a positive ground state solution u_ε such that $\Phi_\varepsilon(u_\varepsilon)=c_\varepsilon$. Now, we take $\varepsilon_n\downarrow 0$ as $n\to+\infty$ and assume that u_n is a positive ground state solution of (4.2) with $\varepsilon=\varepsilon_n$. Similar to the formula (4.16), it is easy to prove that*

$$c_{\varepsilon_n}=\Phi_{\varepsilon_n}(u_n)\geq(\frac{1}{2}-\frac{1}{2^*(s_k)})\|u_n\|^2.\tag{6.1}$$

By Lemma 4.10, we see that $\limsup\limits_{n\to+\infty}c_{\varepsilon_n}\leq c_0$. Hence, $\{u_n\}$ is bounded in $D_0^{1,2}(\mathbb{R}^N)$. Up to a subsequence, we assume that $u_n\rightharpoonup u_0$ in $D_0^{1,2}(\mathbb{R}^N)$ and $u_n\to u_0$ a.e. in $\mathbb{R}^N$.

Lemma 6.1. *u_0 is a critical point of Φ_0, i.e., $\Phi_0'(u_0)=0$.*

Proof. We claim that for any $\phi \in D_0^{1,2}(\mathbb{R}^N)$ and $i \in \{1,2,\cdots,l\}$, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}\left[a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)-2}u_n\phi\right]dx=\int_{\mathbb{R}^N}\left[\frac{1}{|x|^{s_i}}|u_0|^{2^*(s_i)-2}u_0\phi\right]dx. \tag{6.2}$$

Without loss of generality, we may also assume that $\phi \geq 0$. Otherwise, we write $\phi = \phi_+ - \phi_-$ and discuss ϕ_+ and ϕ_-, respectively. Firstly by the Fatou's Lemma, it is easy to see that

$$\int_{\mathbb{R}^N}\left[\frac{1}{|x|^{s_i}}|u_0|^{2^*(s_i)-2}u_0\phi\right]dx\leq\liminf_{n\to+\infty}\int_{\mathbb{R}^N}\left[a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)-2}u_n\phi\right]dx. \tag{6.3}$$

On the other hand, since $a_{i,\varepsilon_n}(x) \leq a_{i,0}(x) = \frac{1}{|x|^{s_i}}$, we have

$$\int_{\mathbb{R}^N}\left[a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)-2}u_n\phi\right]dx\leq\int_{\mathbb{R}^N}\left[\frac{1}{|x|^{s_i}}|u_n|^{2^*(s_i)-2}u_n\phi\right]dx. \tag{6.4}$$

Since $u_n \rightharpoonup u_0$ in $D_0^{1,2}(\mathbb{R}^N)$, we see that

$$|u_n|^{2^*(s_i)-2}u_n \rightharpoonup |u_0|^{2^*(s_i)-2}u_0 \text{ in } L^{\frac{2^*(s_i)}{2^*(s_i)-1}}(\mathbb{R}^N).$$

Hence, we have

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}\left[\frac{1}{|x|^{s_i}}|u_n|^{2^*(s_i)-2}u_n\phi\right]dx=\int_{\mathbb{R}^N}\left[\frac{1}{|x|^{s_i}}|u_0|^{2^*(s_i)-2}u_0\phi\right]dx. \tag{6.5}$$

By (6.4) and (6.5), we obtain

$$\limsup_{n\to+\infty}\int_{\mathbb{R}^N}\left[a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)-2}u_n\phi\right]dx\leq\int_{\mathbb{R}^N}\left[\frac{1}{|x|^{s_i}}|u_0|^{2^*(s_i)-2}u_0\phi\right]dx. \tag{6.6}$$

Thus, (6.2) is proved by (6.3) and (6.6). Recalling that u_n is a critical point of Φ_{ε_n}, we have that $\langle \Phi'_{\varepsilon_n}(u_n), \phi\rangle = 0$ for all $\phi \in D_0^{1,2}(\mathbb{R}^N)$. Then by (6.2) and $u_n \rightharpoonup u_0$ in $D_0^{1,2}(\mathbb{R}^N)$, we see that $\langle \Phi'_0(u_0), \phi\rangle = 0$ for all $\phi \in D_0^{1,2}(\mathbb{R}^N)$, i.e., $\Phi'_0(u_0) = 0$. □

Lemma 6.2. $0 \leq \Phi_0(u_0) \leq \lim\limits_{n\to+\infty} c_{\varepsilon_n} \leq c_0$ *and if* $u_0 \neq 0$*, we have* $\Phi_0(u_0) = c_0 > 0$.

Proof. By Lemma 6.1, $\Phi'_0(u_0) = 0$. If $u_0 = 0$, we have $\Phi_0(u_0) = 0$. If $u_0 \neq 0$, it is easy to see that $u_0 \in \mathcal{N}_0$, then it follows that $\Phi_0(u_0) \geq c_0 > 0$. Hence, we always have

$$\Phi_0(u_0) \geq 0. \tag{6.7}$$

Since u_n is a ground state solution of (4.2) with $\varepsilon = \varepsilon_n$, similar to (4.16), we have that

$$\begin{aligned} c_{\varepsilon_n} = \Phi_{\varepsilon_n}(u_n) &= \left[\frac{1}{2}-\frac{1}{2^*(s_k)}\right]\|u_n\|^2+\sum_{i=1}^{l}\left[\frac{1}{2^*(s_k)}-\frac{1}{2^*(s_i)}\right]\lambda_i|u_n|^{2^*(s_i)}_{2^*(s_i),i,\varepsilon_n} \\ &\quad+\left[\frac{1}{2^*(s_k)}-\frac{1}{2^*}\right]|u_n|^{2^*}_{2^*}. \end{aligned} \tag{6.8}$$

Noting that $\left[\dfrac{1}{2^*(s_k)} - \dfrac{1}{2^*(s_i)}\right]\lambda_i > 0$ for $i \neq k$, then by Fatou's Lemma and Lemma 4.10, we get that

$$\Phi_0(u_0) \leq \liminf_{n\to+\infty} c_{\varepsilon_n} \leq c_0. \tag{6.9}$$

Furthermore, if $u_0 \neq 0$, then by the definition of c_0, it is trivial to obtain the reverse inequality $\Phi_0(u_0) \geq c_0$. Hence, $\Phi_0(u_0) = c_0$. Evidently, $c_0 > 0$, see also Remark 4.1 and Lemma 4.3. □

Lemma 6.3. *Assume that* $\lim\limits_{n\to+\infty}\int_{\mathbb{R}^N} a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)}dx = 0$ *for some* $i \in \{1, 2, \cdots, l\}$*, then* $\{u_n\}$ *is a PS sequence of* Ψ*, i.e.,* $\Psi'(u_n) \to 0$*.*

Proof. Noting that $\{u_n\}$ is bounded in $D_0^{1,2}(\mathbb{R}^N)$. By Corollary 5.1, we indeed obtain that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N} a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)}dx = 0 \text{ for all } i = 1, 2, \cdots, l. \tag{6.10}$$

Then by Hölder inequality and Hardy-Sobolev inequality, we see that

$$\int_{\mathbb{R}^N}\Big[\sum_{i=1}^{l}\lambda_i a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)-2}u_n h\Big]dx = o(1)\|h\|. \tag{6.11}$$

Recalling that $\Phi'_{\varepsilon_n}(u_n) = 0$, we obtain that

$$\langle\Psi'(u_n), h\rangle \equiv \int_{\mathbb{R}^N}\Big[\sum_{i=1}^{l}\lambda_i a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)-2}u_n h\Big]dx = o(1)\|h\|. \tag{6.12}$$

Hence, $\Psi'(u_n) \to 0$. □

Corollary 6.1. $\lim\limits_{n\to+\infty}\int_{\mathbb{R}^N}\Big[a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)}\Big]dx > 0$ *for all* $i = 1, 2, \cdots, l$*.*

Proof. We prove it by the way of negation. We assume that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}\Big[a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)}\Big]dx = 0 \text{ for some } i \in \{1, 2, \cdots, l\}. \tag{6.13}$$

Then by Lemma 6.3, $\{u_n\}$ is a PS sequence of Ψ. By Remark 4.1, we always have $\liminf\limits_{n\to+\infty} c_{\varepsilon_n} > 0$. Hence, $u_n \not\to 0$ in $D_0^{1,2}(\mathbb{R}^N)$, and then it is easy to see that

$$\lim_{n\to+\infty}\Psi(u_n) \geq \frac{1}{N}S^{\frac{N}{2}}. \tag{6.14}$$

We note that under the assumption (6.13), one can easily obtain that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}\Big[\sum_{i=1}^{l}\frac{1}{2^*(s_i)}\lambda_i a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)}\Big]dx = 0. \tag{6.15}$$

Thus, up to a subsequence, we can obtain that

$$\lim_{n\to+\infty}\Phi_{\varepsilon_n}(u_n)=\lim_{n\to+\infty}\left[\Psi(u_n)-\int_{\mathbb{R}^N}\Big[\sum_{i=1}^{l}\frac{1}{2^*(s_i)}\lambda_i a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)}\Big]dx\right] \tag{6.16}$$

and the above limit is $\geq \frac{1}{N}S^{\frac{N}{2}}$, a contradiction to Lemma 4.8. $\square$

Lemma 6.4. *Let $\varepsilon_n \downarrow 0$. Assume that $\{\phi_n\}\subset D_0^{1,2}(\mathbb{R}^N)$ is a bounded sequence such that*

$$\lim_{n\to+\infty} J_{\varepsilon_n}(\phi_n)=0 \tag{6.17}$$

and $u_n \not\to 0$ in $L^{2^*}(\mathbb{R}^N)$. Suppose that there exists some $i\in\{1,2,\cdots,l\}$ such that

$$\liminf_{n\to+\infty}\int_{\mathbb{R}^N}\Big[a_{i,\varepsilon_n}(x)|\phi_n|^{2^*(s_i)}\Big]dx>0, \tag{6.18}$$

Then up to a subsequence, we must have

$$\lim_{n\to+\infty}\Phi_{\varepsilon_n}(\phi_n)\geq \lim_{n\to+\infty} c_{\varepsilon_n}>0. \tag{6.19}$$

Proof. Up to a subsequence if necessary, we denote

$$\eta_i:=\liminf_{n\to+\infty}\int_{\mathbb{R}^N}\lambda_i a_{i,\varepsilon_n}(x)|\phi_n|^{2^*(s_i)}dx. \tag{6.20}$$

Obviously, $\phi_n \not\to 0$ in $D_0^{1,2}(\mathbb{R}^N)$. If not, by the Sobolev inequality we obtain that $u_n\to 0$ in $L^{2^*}(\mathbb{R}^n)$, a contradiction. Since also that $\{u_n\}$ is bounded in $D_0^{1,2}(\mathbb{R}^N)$, up to a subsequence, there exists some $d_1,d_2>0$ such that

$$0<d_1\leq \|u_n\|^2\leq d_2. \tag{6.21}$$

By the Brézis-Lieb lemma, $u_n\not\to 0$ in $L^{2^*}(\mathbb{R}^N)$ yields that there exist some $d_3>0$ such that

$$d_3\leq |\phi_n|_{2^*}^{2^*}. \tag{6.22}$$

On the other hand, by the Sobolev inequality again, there exists some $d_4>0$ such that

$$|\phi_n|_{2^*}^{2^*}\leq d_4. \tag{6.23}$$

Now, up to a subsequence, we may assume that

$$\|\phi_n\|^2\to a^*>0, |\phi_n|_{2^*}^{2^*}\to b^*>0. \tag{6.24}$$

Then by the assumption (6.17), we have that

$$a^*-\sum_{i=1}^{l}\eta_i-b^*=0. \tag{6.25}$$

If there exists some $i \in \{1, 2, \cdots, l\}$ such that (6.18), then by Corollary 5.1, we obtain that (6.18) holds for all $i \in \{1, 2, \cdots, l\}$. On the other hand, by Lemma 4.3, for ϕ_n, there exists a unique $t_n > 0$ such that $t_n\phi_n \in \mathcal{N}_{\varepsilon_n}$. Hence,

$$\|\phi_n\|^2 - \sum_{i=1}^{l} \lambda_i a_{i,\varepsilon_n}(x)|\phi_n|^{2^*(s_i)} t_n^{2^*(s_i)-2} - |\phi_n|_{2^*}^{2^*} t_n^{2^*-2} = 0. \tag{6.26}$$

Then firstly we have

$$\begin{aligned}\|\phi_n\|^2 =& \sum_{i=1}^{l} \lambda_i a_{i,\varepsilon_n}(x)|\phi_n|^{2^*(s_i)} t_n^{2^*(s_i)-2} + |\phi_n|_{2^*}^{2^*} t_n^{2^*-2} \\ \leq& C_i|\lambda_i|\|\phi_n\|^{2^*(s_i)} t_n^{2^*(s_i)-2} + C\|\phi_n\|^{2^*} t_n^{2^*-2}.\end{aligned} \tag{6.27}$$

We claim that t_n is bounded away from 0. If not, we assume that $t_n \to 0$, then since $\|\phi_n\| \leq \sqrt{d_2}$, the right hand side of (6.27) goes to 0. But by (6.21), the left hand side of (6.27) is lager than $d_1 > 0$, we obtain a contradiction. Secondly, by (6.22) and (6.26), it is easy to see that $\{t_n\}$ is bounded. Hence, we may assume that $t_n \to t^* > 0$. Then we have

$$\left.\begin{array}{l} J_{\varepsilon_n}(t_n\phi_n) \equiv 0, \\ \{\phi_n\} \text{ is bounded in } D_0^{1,2}(\mathbb{R}^N), \end{array}\right\} \Rightarrow \lim_{n\to+\infty} J_{\varepsilon_n}(t^*\phi_n) = 0.$$

Then it follows that

$$a^* - \sum_{i=1}^{l} \eta_i (t^*)^{2^*(s_i)-2} - b^*(t^*)^{2^*-2} = 0. \tag{6.28}$$

Apply the similar arguments of Lemma 4.3, we can prove that the algebraic equation $a^* - \sum_{i=1}^{l} \eta_i t^{2^*(s_i)-2} - b^* t^{2^*-2} = 0$ has an unique positive solution. Hence, by (6.25) and (6.28), we obtain that $t^* = 1$. Then by the boundedness of $\{\phi_n\}$ again, it is easy to see that

$$\lim_{n\to+\infty} \Phi_{\varepsilon_n}(\phi_n) = \lim_{n\to+\infty} \Phi_{\varepsilon_n}(t_n\phi_n). \tag{6.29}$$

By the definition of t_n, we see that $t_n\phi_n \in \mathcal{N}_{\varepsilon_n}$. Hence, $\Phi_{\varepsilon_n}(t_n\phi_n) \geq c_{\varepsilon_n}$. It follows that

$$\lim_{n\to+\infty} \Phi_{\varepsilon_n}(\phi_n) \geq \lim_{n\to+\infty} c_{\varepsilon_n}. \tag{6.30}$$

Insert Remark 4.1 here, we have that $\lim\limits_{n\to+\infty} c_{\varepsilon_n} > 0$. $\square$

6.2 The proof of the existence result of Theorem 1.1 for $k = l$

Let ε_n and u_n be defined by Remark 6.1. By Lemma 6.2, we only need to prove that $u_0 \neq 0$. Now, we will proceed by contradiction. We assume that $u_0 = 0$. By Corollary 6.1,

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} \Big[a_{i,\varepsilon_n}(x)|u_n|^{2^*(s_i)}\Big] dx > 0 \text{ for all } i = 1, 2, \cdots, l.$$

Recalling that $\{u_n\}$ is bounded and all $\lambda_i s$ are positive, up to a subsequence, we can denote that

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}\Big[\sum_{i=1}^{l}\frac{\lambda_i a_{i,\varepsilon_n}(x)}{2^*(s_i)}|u_n|^{2^*(s_i)}\Big]dx =: \tau > 0. \tag{6.31}$$

Thus, by Corollary 5.2, we obtain that

$$\begin{aligned}
&\lim_{n\to+\infty}\int_{\mathbb{B}_1}\Big[\sum_{i=1}^{l}\frac{\lambda_i a_{i,\varepsilon_n}(x)}{2^*(s_i)}|u_n|^{2^*(s_i)}\Big]dx\\
&=\lim_{n\to+\infty}\int_{\mathbb{B}_1^c}\Big[\sum_{i=1}^{l}\frac{\lambda_i a_{i,\varepsilon_n}(x)}{2^*(s_i)}|u_n|^{2^*(s_i)}\Big]dx\\
&=\frac{\tau}{2}>0.
\end{aligned}\tag{6.32}$$

Let $\chi(x)\in C_c^\infty(\mathbb{R}^N)$ be a cut-off function such that $\chi(x)\equiv 1$ in $\mathbb{B}_{\frac{1}{2}}$, $\chi(x)\equiv 0$ in $\mathbb{R}^N\backslash\mathbb{B}_1$ and take $\tilde{\chi}(x)\in C^\infty(\mathbb{R}^N)$ such that $\tilde{\chi}(x)\equiv 0$ in $\mathbb{B}_1$ and $\tilde{\chi}\equiv 1$ in $\mathbb{R}^N\backslash\mathbb{B}_2$. Let us denote

$$\phi_{1,n}(x) := \chi(x)u_n(x), \phi_{2,n}(x) := \tilde{\chi}(x)u_n(x) \tag{6.33}$$

and define

$$\tilde{u}_n := u_n - \phi_{1,n} - \phi_{2,n}. \tag{6.34}$$

Then we see that $sppt(\tilde{u}_n)\subset\Omega$, where $\Omega := \{x\in\mathbb{R}^N : \frac{1}{2}<|x|<2\}$. Then by the Rellich-Kondrachov compactness theorem, we see that $\tilde{u}_n\to 0$ strongly in $L^{2^*(s_i)}(\Omega,\frac{dx}{|x|^{s_i}})$ for all $i=1,2,\cdots,l$. Then it follows that $\tilde{u}_n$ is a PS sequence of Ψ. By Brézis-Lieb Lemma, we can prove that

$$\Phi_{\varepsilon_n}(u_n) = \Phi_{\varepsilon_n}(\phi_{1,n}+\phi_{2,n}) + \Psi(\tilde{u}_n) + o(1). \tag{6.35}$$

Recalling that $\Phi'_{\varepsilon_n}(u_n)\equiv 0$, it is easy to prove that

$$\lim_{n\to+\infty}\Phi'_{\varepsilon_n}(\phi_{1,n}+\phi_{2,n}) = 0. \tag{6.36}$$

Obviously,

$$\lim_{n\to+\infty}\Phi_{\varepsilon_n}(\phi_{1,n}+\phi_{2,n}) \ge 0. \tag{6.37}$$

Hence, if $\tilde{u}_n\not\to 0$, we have that

$$\lim_{n\to+\infty}\Psi(\tilde{u}_n) \ge \frac{1}{N}S^{\frac{N}{2}}. \tag{6.38}$$

By (6.36), (6.37) and (6.35), we obtain that $\lim\limits_{n\to+\infty} c_{\varepsilon_n} \ge \frac{1}{N}S^{\frac{N}{2}}$, a contradiction to Lemma 4.8. Hence, we prove that $\tilde{u}_n\to 0$ in $D_0^{1,2}(\mathbb{R}^N)$ and it follows that

$$\Phi_{\varepsilon_n}(u_n) = \Phi_{\varepsilon_n}(\phi_{1,n}) + \Phi_{\varepsilon_n}(\phi_{2,n}) + o(1). \tag{6.39}$$

Recal that $\Phi'_{\varepsilon_n}(u_n) \equiv 0$ and hence $\langle \Phi'_{\varepsilon_n}(u_n), \phi_{1,n}\rangle \equiv 0$. Then by $\tilde{u}_n \to 0$ strongly in $D_0^{1,2}(\mathbb{R}^N)$, it is easy to see that

$$\lim_{n\to+\infty} J_{\varepsilon_n}(\phi_{1,n}) = 0. \tag{6.40}$$

By (6.32) and the Rellich-Kondrachov compactness result, we can prove that

$$\begin{aligned}
&\liminf_{n\to+\infty} \int_{\mathbb{R}^N} \Big[\sum_{i=1}^{l} \frac{\lambda_i a_{i,\varepsilon_n}(x)}{2^*(s_i)} |\phi_{1,n}|^{2^*(s_i)}\Big] dx \\
&= \lim_{n\to+\infty} \int_{\mathbb{B}_1} \Big[\sum_{i=1}^{l} \frac{\lambda_i a_{i,\varepsilon_n}(x)}{2^*(s_i)} |u_n|^{2^*(s_i)}\Big] dx \\
&= \frac{\tau}{2} > 0.
\end{aligned} \tag{6.41}$$

And it follows easily that

$$\liminf_{n\to+\infty} \int_{\mathbb{R}^N} \Big[a_{i,\varepsilon_n}(x) |\phi_{1,n}|^{2^*(s_i)}\Big] dx > 0 \text{ for all } i = 1, 2, \cdots, l. \tag{6.42}$$

Hence, by (6.40), (6.41), (6.42) and Lemma 6.4, we obtain that

$$\liminf_{n\to+\infty} \Phi_{\varepsilon_n}(\phi_{1,n}) \geq \lim_{n\to+\infty} c_{\varepsilon_n}. \tag{6.43}$$

Similarly, we can also obtain that

$$\liminf_{n\to+\infty} \Phi_{\varepsilon_n}(\phi_{2,n}) \geq \lim_{n\to+\infty} c_{\varepsilon_n}. \tag{6.44}$$

Hence, by (6.39), (6.43) and (6.44), we have that $\lim_{n\to+\infty} c_{\varepsilon_n} \geq 2 \lim_{n\to+\infty} c_{\varepsilon_n}$, it is a contradiction to the fact of that $\lim_{n\to+\infty} c_{\varepsilon_n} > 0$ because of Lemma 6.4. Thereby $u_0 \neq 0$ is proved. □

6.3 The proof of the existence result of Theorem 1.1 for $k \neq l$

When $k \neq l$, the proof becomes very thorny and we have to apply another way-perturbation methods. In this case, we assume that $l \geq 2$. For the convenience, in this subsection we denote

$$I_0(u) = I(u) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 \, dx - \int_{\mathbb{R}^N} \sum_{i=1}^{k} \frac{\lambda_i}{2^*(s_i)} \frac{|u|^{2^*(s_i)}}{|x|^{s_i}}, \ u \in D_0^{1,2}(\mathbb{R}^N), \tag{6.45}$$

and

$$I_\lambda(u) = I_0(u) - \lambda \int_{\mathbb{R}^N} \left(\sum_{i=k+1}^{l} \frac{1}{2^*(s_i)} |\lambda_i| \int_{\mathbb{R}^N} \frac{|u|^{2^*(s_i)}}{|x|^{s_i}}\right), \ u \in D_0^{1,2}(\mathbb{R}^N), \tag{6.46}$$

which is the corresponding functional of the following variant problem:

$$\begin{cases}\Delta u + u^{2^*-1} + \sum_{i=1}^{k} \lambda_i \frac{u^{2^*(s_i)-1}}{|x|^{s_i}} + \lambda \left(\sum_{i=k+1}^{l} |\lambda_i| \frac{u^{2^*(s_i)-1}}{|x|^{s_i}}\right) = 0, \ x \in \ \mathbb{R}^N, \\ u \in D_0^{1,2}(\mathbb{R}^N). \end{cases} \tag{6.47}$$

We note that when $\lambda = -1$, it becomes the problem (1.3). Hence, our aim is to prove that problem (6.47) possesses a least energy solution when $\lambda = -1$. We set

$$D_k := \{\mu \in \mathbb{R} \ : \ \text{problem (6.47) possesses a least energy solution when } \lambda = \mu\}. \tag{6.48}$$

Then we only need to prove that $-1 \in D_k$. Firstly, by the results established in the previous subsection, for any $\lambda \geq 0$, problem (6.47) possesses a least energy solution, and the corresponding energy is less than $\frac{1}{N} S^{\frac{N}{2}}$. Hence,

$$[0, +\infty) \subset D_k. \tag{6.49}$$

Set

$$\mathcal{A}_\mu := \left\{u \in D_0^{1,2}(\mathbb{R}^N) \text{ is a positive solution of problem (6.47) when } \lambda = \mu\right\}. \tag{6.50}$$

When $\mathcal{A}_\lambda \neq \emptyset$, we define

$$c_\lambda^* := \inf_{u \in \mathcal{A}_\lambda} I_\lambda(u), \tag{6.51}$$

$$c_\lambda := \inf_{u \in D_0^{1,2}(\mathbb{R}^N) \setminus \{0\}} \max_{t > 0} I_\lambda(tu) \tag{6.52}$$

and

$$\tilde{c}_\lambda := \min_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(\gamma(t)), \tag{6.53}$$

where $\Gamma_\lambda := \left\{\gamma \in C\left([0,1], D_0^{1,2}(\mathbb{R}^N)\right) : \ \gamma(0) = 0, I_\lambda(\gamma(1)) < 0\right\}$. Then it is easy to see that

$$c_\lambda^* \geq c_\lambda \geq \tilde{c}_\lambda > 0. \tag{6.54}$$

By the standard concentration compactness arguments, one can prove that if there exists a bounded $(PS)_c$ sequence of I_λ with $c < c_\lambda^*$, then c is a critical value of I_λ. It follows that

$$c_\lambda^* = c_\lambda = \tilde{c}_\lambda > 0. \tag{6.55}$$

Next, we prepare the following properties about the least energy solution.

Lemma 6.5. *Assume that for some $\lambda \in \mathbb{R}$, equation* (6.47) *possesses a least energy solution $u_\lambda(x)$, then*

$$0 < I_\lambda(u_\lambda) = c_\lambda < \frac{1}{N} S^{\frac{N}{2}}. \tag{6.56}$$

On the other hand, the functional I_λ possesses the following properties:

(M1) there exists some $c, r > 0$ such that $I_\lambda(u) \geq c$ for $\|u\| = r$. Moreover, there exists $v_\lambda \in D_0^{1,2}(\mathbb{R}^N)$ such that $\|v_\lambda\| > r$ and $I_\lambda(v_\lambda) < 0$;

(M2) there exists a critical point $u_\lambda \in D_0^{1,2}(\mathbb{R}^N)$ of I_λ such that

$$I_\lambda(u_\lambda) = c_\lambda = \tilde{c}_\lambda := \min_{\gamma \in \Gamma} \max_{t \in [0,1]} I_\lambda(\gamma(t)), \tag{6.57}$$

where $\Gamma_\lambda := \left\{\gamma \in C\left([0,1], D_0^{1,2}(\mathbb{R}^N)\right) : \gamma(0) = 0, \gamma(1) = v_\lambda\right\}$;

(M3) $c_\lambda = c_\lambda^* = \inf\left\{I_\lambda(u) : I'_\lambda(u) = 0, u \in D_0^{1,2}(\mathbb{R}^N)\backslash\{0\}\right\}$

(M4) the set $\mathcal{S}_\lambda := \left\{u \in D_0^{1,2}(\mathbb{R}^N) : I'_\lambda(u) = 0, I_\lambda(u) = c_\lambda\right\}$ is compact in $D_0^{1,2}(\mathbb{R}^N)$;

(M5) there exists a path $\gamma_\lambda(t) \in \Gamma_\lambda$ passing through u_λ at $t = t_\lambda$ and satisfying

$$I_\lambda(u_\lambda) > I_\lambda(\gamma_\lambda(t)) \text{ for all } t \neq t_\lambda. \tag{6.58}$$

Proof. Obviously, $c_\lambda > 0$. Combining with the result of Lemma 4.8, we obtain (6.56) and (6.52). Based on the result of (6.55), (M1)-(M3) are trivial. And by Lemma 4.3 we can obtain (M5). Hence, next we only need to check the property of (M4). Let $\{u_n\} \subset \mathcal{S}_\lambda$, noting that $I'_\lambda(u_n) = 0$, by Lemma 4.4 we see that $\{u_n\}$ is a bounded $(PS)_{c_\lambda}$ sequence of I_λ. And it is easy to prove that $|u_n|_{2^*}$ are bounded away from 0. On the other hand, by the results of section 3, u_n is radial and decreasing by $|x|$. Also by Proposition 2.1, we see that $\{u_n(0)\}$ is bounded. Hence, $u_n(x)$ is a bounded sequence of $L^\infty(\mathbb{R}^N) \cap D_{rad}^{1,2}(\mathbb{R}^N)$, where $D_{rad}^{1,2}(\mathbb{R}^N)$ is the radial subspace of $D_0^{1,2}(\mathbb{R}^N)$. Noting that the Kelvin transform of u_n, which is denoted by $\hat{u}_n(x) := |x|^{-(N-2)} u_n\left(\frac{x}{|x|^2}\right)$, is also a least energy solution, i.e., $\hat{u}_n \in \mathcal{S}_\lambda$. On the other hand, we also note that for any $s \in [0,2]$ and any solution u with its Kelvin transform $\hat{u}$, we have

$$\int_{\mathbb{B}_1} \frac{|u|^{2^*(s)}}{|x|^s} dx = \int_{\mathbb{B}_1^c} \frac{|\hat{u}|^{2^*(s)}}{|x|^s} dx. \tag{6.59}$$

Hence, for the new sequence $\{u_1, v_1, u_2, v_2, \cdots\} \subset \mathcal{S}_\lambda$, there exists a subsequence, denoted by w_j, such that

$$\liminf_{j \to \infty} \int_{\mathbb{B}_1} |w_j|^{2^*} dx > 0. \tag{6.60}$$

Then by $u_n(0) = \sup_{x \in \mathbb{R}^N} u_n(x)$(see Remark 3.2), we obtain that

$$\liminf_{j \to +\infty} w_j(0) > 0, \tag{6.61}$$

Hence, up to a subsequence, $w_j \rightharpoonup w \neq 0$ in $D_0^{1,2}(\mathbb{R}^N)$. It is easy to see that w is also a critical point of I_λ. Hence, we have

$$I_\lambda(w) \geq c_\lambda. \tag{6.62}$$

On the other hand, by the weak semi-continuous of a norm, when $\lambda \le 0$,

$$
\begin{aligned}
I_\lambda(w) = &\left[\frac{1}{2} - \frac{1}{2^*(s_k)}\right]\|w\|^2 + \sum_{i=1}^{k-1}\left[\frac{1}{2^*(s_k)} - \frac{1}{2^*(s_i)}\right]\lambda_i |w|_{2^*(s_i)}^{2^*(s_i)} \\
&+ \sum_{i=k+1}^{l}\lambda\left[\frac{1}{2^*(s_k)} - \frac{1}{2^*(s_i)}\right]|\lambda_i||w|_{2^*(s_i)}^{2^*(s_i)} + \left[\frac{1}{2^*(s_k)} - \frac{1}{2^*}\right]|w|_{2^*}^{2^*} \qquad (6.63)\\
\le &\liminf_{j\to\infty}\left\{\left[\frac{1}{2} - \frac{1}{2^*(s_k)}\right]\|w_j\|^2 + \sum_{i=1}^{k-1}\left[\frac{1}{2^*(s_k)} - \frac{1}{2^*(s_i)}\right]\lambda_i |w_j|_{2^*(s_i)}^{2^*(s_i)}\right. \\
&\left.+ \sum_{i=k+1}^{l}\lambda\left[\frac{1}{2^*(s_k)} - \frac{1}{2^*(s_i)}\right]|\lambda_i||w_j|_{2^*(s_i)}^{2^*(s_i)} + \left[\frac{1}{2^*(s_k)} - \frac{1}{2^*}\right]|w_j|_{2^*}^{2^*}\right\} \qquad (6.64)\\
= &\liminf_{i\to\infty} I_\lambda(w_j) = c_\lambda. \qquad (6.65)
\end{aligned}
$$

The case of $\lambda > 0$ is much easier to check. Then, it follows that $I_\lambda(w) = c_\lambda$ and thus $w_j \to w \in \mathcal{S}_\lambda$, its Kelvin transform $\hat{w}$ also satisfies $\hat{w} \in \mathcal{S}_\lambda$. Hence, up to a subsequence, we have $u_n \to w \in \mathcal{S}_\lambda$ or $u_n \to \hat{w} \in \mathcal{S}_\lambda$. Thereby, (M4) is verified. □

Remark 6.2. *By Remark 3.2, we see that every u_i, v_i above are radial and decreasing by $|x|$. Hence, if the sequence concentrated, it can only happen at $x = 0$, i.e.,*

$$\sup_{y\in\mathbb{R}^N}\int_{\mathbb{B}_1(y)}|u_i|^{2^*}dx \equiv \int_{\mathbb{B}_1(0)}|u_i|^{2^*}dx, \qquad (6.66)$$

so does v_i.

Lemma 6.6. *For any $\lambda \in D_k$, there exists some $\delta_\lambda > 0$ small enough such that $(\lambda - \delta_\lambda, \lambda + \delta_\lambda) \subset D_k$. In other words, D_k is an open set of $\mathbb{R}$.*

Proof. Basing on the Lemma 6.5, applying the perturbation arguments, it is standard to prove the existence of δ_λ and the existence of positive solution for $\mu \in (\lambda - \delta_\lambda, \lambda + \delta_\lambda)$. This processes is very long and tedious, however it is standard. Hence, we omit the details and a very like discussion we refer to [1, section 5]. Next, we shall prove the existence of least energy solution. For any fixed $\mu \in (\lambda - \delta_\lambda, \lambda + \delta_\lambda)$, then we firstly have that $\mathcal{A}_\mu \neq \emptyset$. Let $\{u_n\} \subset \mathcal{A}_\mu$ be a minimizing sequence, then it is easy to see that $\{u_n\}$ is a bounded $(PS)_{c_\mu^*}$ sequence of I_μ. Let $\hat{u}_n$ be the Kelvin transform of u_n, then we also have that $\{\hat{u}_n\} \subset \mathcal{A}_\mu$. Hence, apply the similar argument of the (M4) in Lemma 6.5, we can prove that $\{u_1, \hat{u}_1, u_2, \hat{u}_2, \cdots\}$ is also a minimizing sequence and it possesses a strong convergent subsequence. Thus, we prove that c_μ^* is achieved. We also note that (6.55) holds. □

Lemma 6.7. *D_k is closed in $\mathbb{R}$.*

Proof. For any sequence $\{\lambda_n\} \subset D_k, \lambda_n \to \lambda$, we shall prove that $\lambda \in D_k$.

Firstly we have

$$\lim_{n\to\infty} c_{\lambda_n} = \lim_{n\to\infty} \tilde{c}_{\lambda_n} = \tilde{c}_\lambda \leq c_\lambda. \tag{6.67}$$

Secondly by Lemma 4.8,

$$0 < \lim_{n\to\infty} c_{\lambda_n} < \frac{1}{N} S^{\frac{N}{2}}. \tag{6.68}$$

Then the boundedness of c_{λ_n} yields the boundedness of $\{u_n\}$ in $D_0^{1,2}(\mathbb{R}^N)$. By $\lambda_n \to \lambda$ and $I'_{\lambda_n}(u_n) = 0$, we see that $I'_\lambda(u_n) \to 0$. Hence, we obtain that $\{u_n\}$ is a bounded, radial, decreasing by $|x|$, $(PS)_{\tilde{c}_\lambda}$ sequence of I_λ. We still adopt the notation $\hat{u}_n$ as the Kelvin transform of u_n, then we firstly have that $I'_{\lambda_n}(\hat{u}_n) = 0$, furthermore, we have that $\{u_1, \hat{u}_1, u_2, \hat{u}_2, \cdots\}$ is also a bounded, radial, decreasing by $|x|$, $(PS)_{\tilde{c}_\lambda}$ sequence of I_λ. Applying the similar argument of the (M4) in Lemma 6.5, we obtain that $\{u_1, \hat{u}_1, u_2, \hat{u}_2, \cdots\}$ possesses a strong convergent subsequence. Hence, up to a subsequence, we may assume that $u_n \to u$ or $u_n \to \hat{u}$. Hence, $\tilde{c}_\lambda$ is achievable, and it follows that $\mathcal{A}_\lambda \neq \emptyset$. Then we have the relationship of (6.55) and thus u is a least energy solution. Hence, $\lambda \in D_k$, and D_k is closed in $\mathbb{R}$. $\square$

The final proof of the existence of least energy solution of Theorem 1.1 for $k \neq l$: By Lemma 6.6 and Lemma 6.7, we see that D_k is a both open and closed set of $\mathbb{R}$. By (6.49), $[0, +\infty) \subset D_k$, hence $D_k \neq \emptyset$. Finally, we obtain that $D_k = \mathbb{R}$. Then $-1 \in D_k$, and thus the existence of Theorem 1.1 is completed. $\square$

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